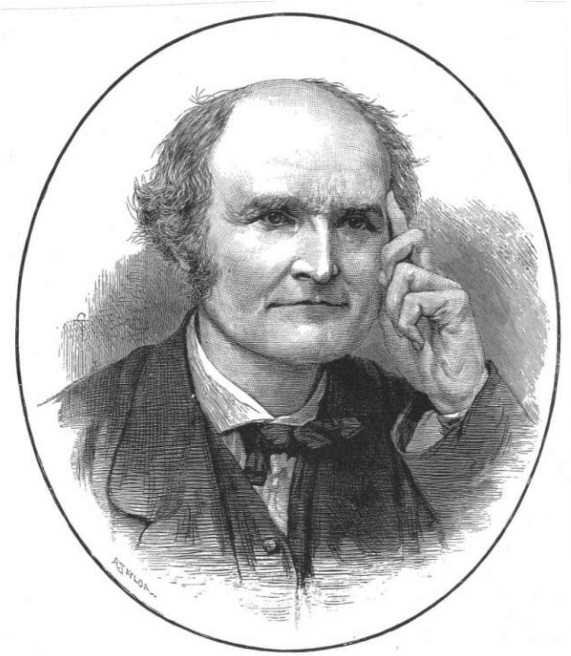
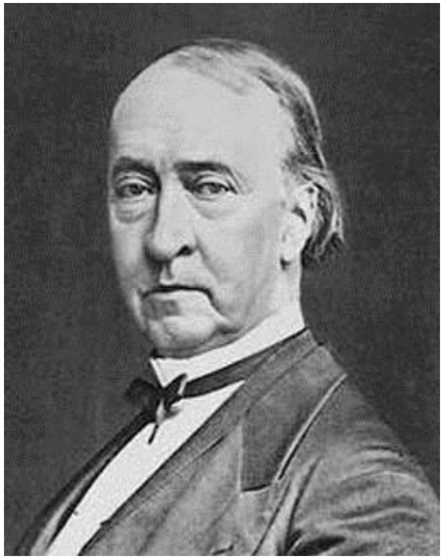
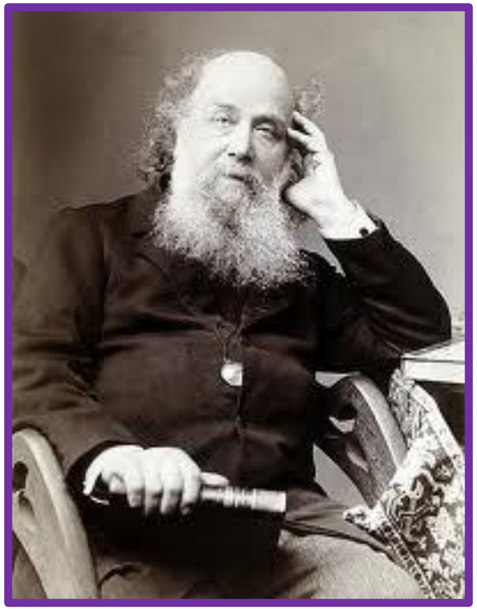


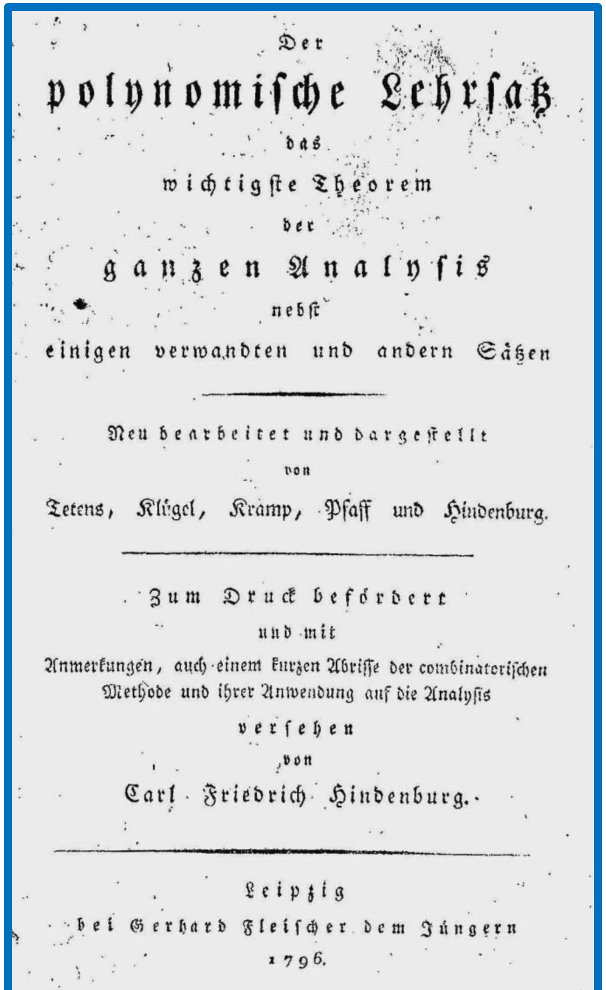
5. The 19th Century



Robin Wilson



Carl Friedrich Hindenburg (1796)



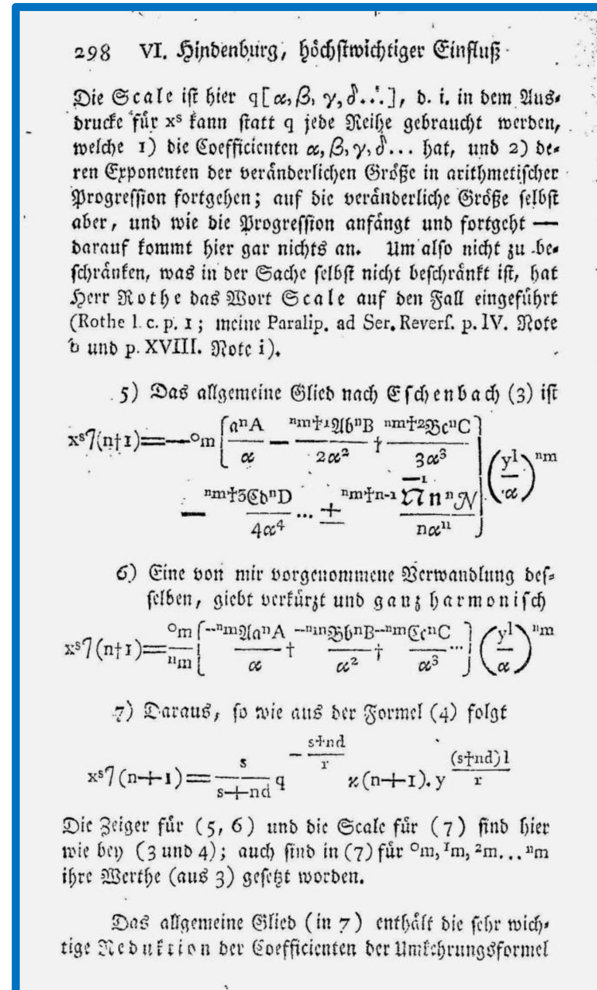
Der
polynomische Lehrsatz
das
wichtigste Theorem
der
ganzen Analysis
nebst
einigen verwandten und andern Sätzen

Neu bearbeitet und dargestellt
von
Tetens, Klügel, Kramp, Pfaff und Hindenburg.

Zum Druck befördert
und mit
Anmerkungen, auch einem kurzen Abrisse der combinatorischen Methode und ihrer Anwendung auf die Analysis
versehen
von
Carl Friedrich Hindenburg.

Leipzig
bei Gerhard Fleischer dem Jüngern
1796.

‘Hindenburg and his school [in Leipzig] attempted through systematic development of combinatorials, to give it a key position within the various mathematical disciplines.’



298 VI. Hindenburg, höchstwichtiger Einfluß

Die Scale ist hier $q[\alpha, \beta, \gamma, \delta \dots]$, d. i. in dem Ausdrucke für x^s kann statt q jede Reihe gebraucht werden, welche 1) die Coefficienten $\alpha, \beta, \gamma, \delta \dots$ hat, und 2) deren Exponenten der veränderlichen Größe in arithmetischer Progression fortgehen; auf die veränderliche Größe selbst aber, und wie die Progression anfängt und fortgeht — darauf kommt hier gar nichts an. Um also nicht zu beschränken, was in der Sache selbst nicht beschränkt ist, hat Herr Rothe das Wort Scale auf den Fall eingeführt (Rothe l. c. p. 1; meine Paralip. ad Ser. Reverf. p. IV. Note b und p. XVIII. Note i).

5) Das allgemeine Glied nach Eschenbach (3) ist

$$x^s\urcorner(n+1) = -{}^0m\left\{\frac{a^nA}{\alpha} - \frac{{}^{nm+1}\mathfrak{A}b^nB}{2\alpha^2} + \frac{{}^{nm+2}\mathfrak{B}c^nC}{3\alpha^3} - \frac{{}^{nm+3}\mathfrak{C}d^nD}{4\alpha^4} \dots \pm \frac{{}^{nm+n-1}\overset{-1}{\mathfrak{N}}n^n\mathcal{N}}{n\alpha^n}\right\}\left(\frac{y^l}{\alpha}\right)^{nm}$$

6) Eine von mir vorgenommene Verwandlung desselben, giebt verkürzt und ganz harmonisch

$$x^s\urcorner(n+1) = \frac{{}^0m}{{}^nm}\left\{\frac{{}^{-nm}\mathfrak{A}a^nA}{\alpha} + \frac{{}^{-nm}\mathfrak{B}b^nB}{\alpha^2} + \frac{{}^{-nm}\mathfrak{C}c^nC}{\alpha^3} \dots\right\}\left(\frac{y^l}{\alpha}\right)^{nm}$$

7) Daraus, so wie aus der Formel (4) folgt

$$x^s\urcorner(n+1) = \frac{s}{s+nd}\, q^{-\frac{s+nd}{r}}\, \varkappa(n+1).\, y^{\frac{(s+nd)l}{r}}$$

Die Zeiger für (5, 6) und die Scale für (7) sind hier wie bey (3 und 4); auch sind in (7) für ${}^0m, {}^1m, {}^2m \dots {}^nm$ ihre Werthe (aus 3) gesetzt worden.

Das allgemeine Glied (in 7) enthält die sehr wichtige Reduktion der Coefficienten der Umkehrungsformel

Peter Nicholson

(1818)

Scottish practical builder and mathematician

ESSAYS

ON THE

COMBINATORIAL ANALYSIS;

SHEWING ITS

APPLICATION

TO SOME OF THE MOST USEFUL AND INTERESTING

Problems of Algebra;

SUCH AS

The Expansion of a Multinomial according to any given Exponent,

THE PRODUCT OF TWO OR MORE MULTINOMIALS,

THE QUOTIENT ARISING BY DIVIDING ONE MULTINOMIAL BY ANOTHER,

The Reversion and Conversion of Series,

THE THEORY OF INDETERMINATE EQUATIONS, &c. &c.

And clearly indicating

THE LAW OF EXPANSION,

AND THE

Simple and almost mechanical Processes from which the resulting Series may be obtained by finding any one Term independently of the Rest.

By P. NICHOLSON,

Private Teacher of the Mathematics.

London:

PRINTED FOR THE AUTHOR,

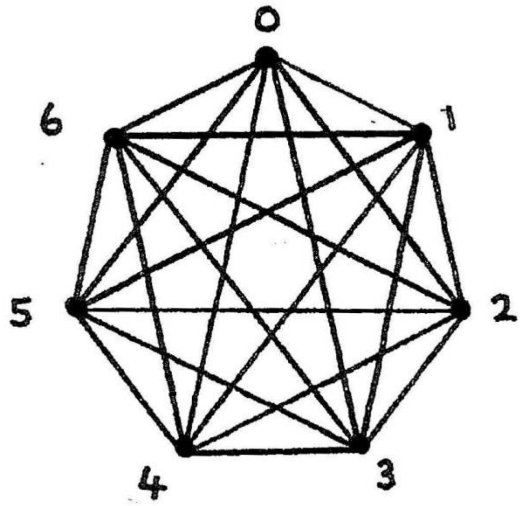
AND PUBLISHED BY LONGMAN, HURST, REES, ORME, AND BROWN, PATERNOSTER ROW; AND BY THE AUTHOR, 12, LONDON STREET, FITZROY SQUARE.

1818.

[Entered at Stationers' Hall.]

Louis Poinsot's diagram-tracing puzzles (1809)

Given some points situated at random in space, it is required to arrange a single flexible thread uniting them two by two in all possible ways, so that finally the two ends of the thread join up, and so that the total length is equal to the sum of all the mutual distances. As we shall see, the solution is possible only for an odd number of points.

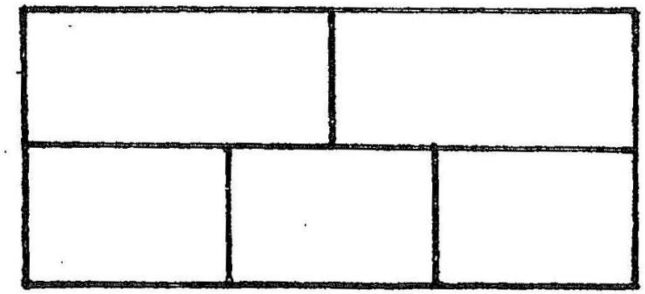


odd complete graphs, K_5, K_7, K_9, . . .;

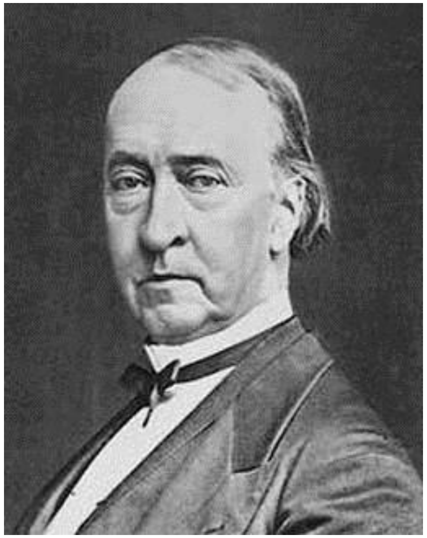
0123456 0246135 03625140

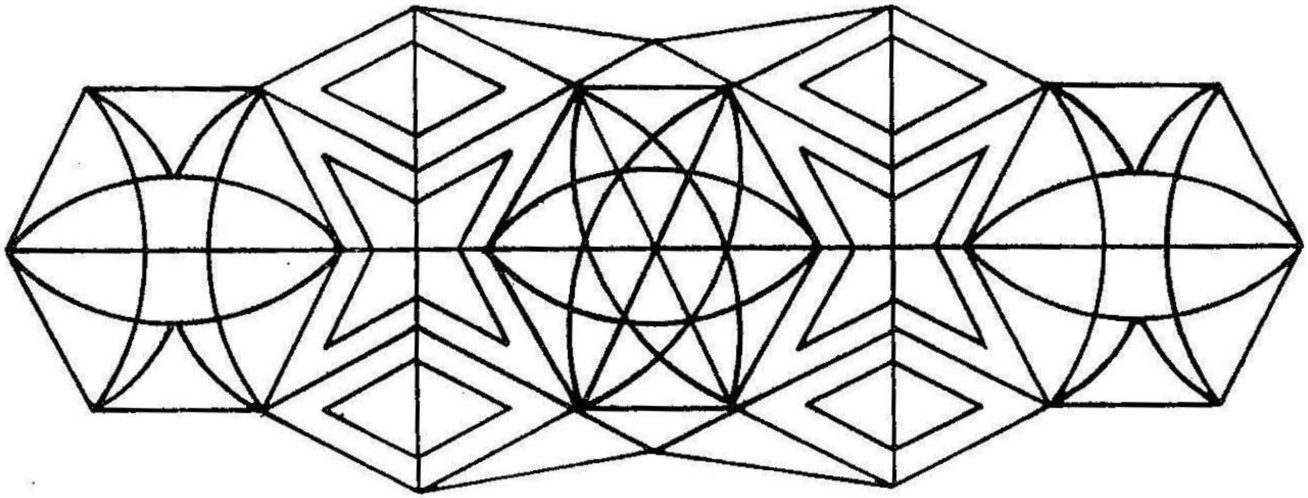
J. B. Listing's diagrams

Vorstudien zur Topologie (1847)



Here there are 8 odd intersections, so we need 4 paths

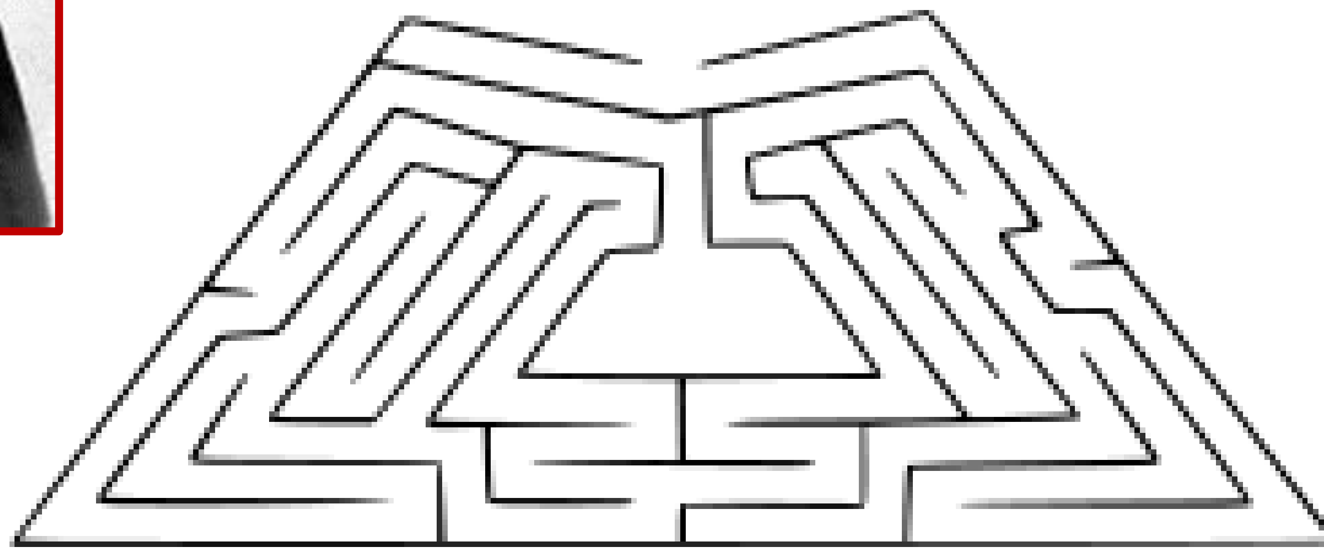




This can be drawn in a single stroke, since it has only two points of odd type, both five-fold . . .

Gaston Tarry

Le problème des labyrinthes (1895)



Tarry's rule: do not return along the passage which has led to a junction for the first time unless you cannot do otherwise.

Tarry also gave a practical method for carrying this out.

W. W. Rouse Ball (1892)

Mathematical Recreations and Problems

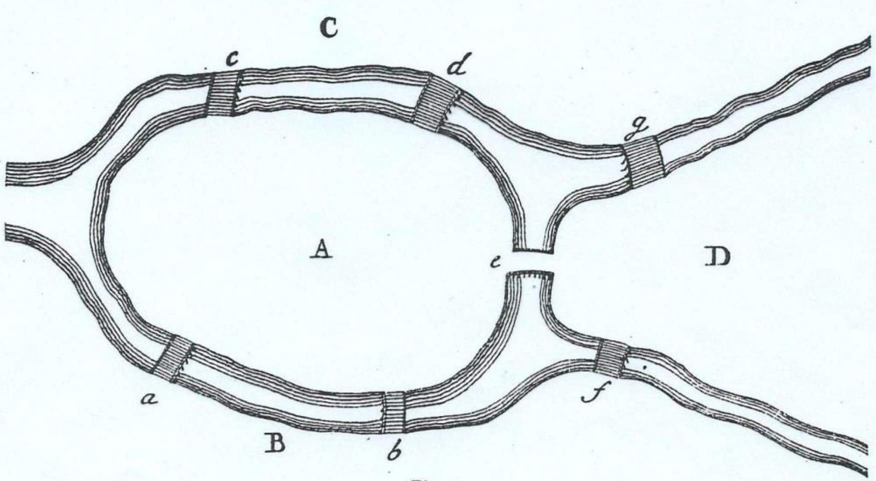
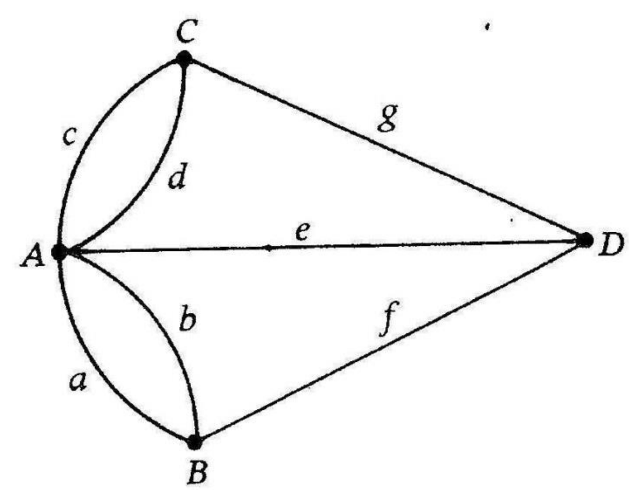


Fig. 1.

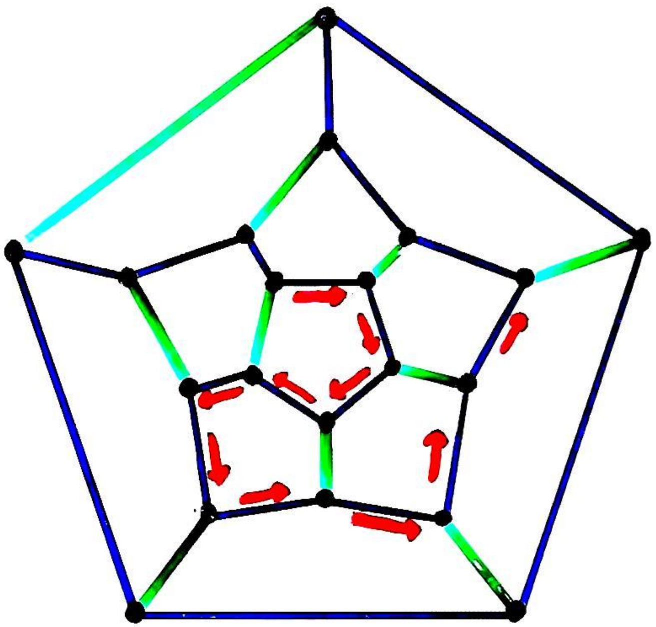


Solving the Königsberg bridges problem corresponds to the solving the diagram-tracing puzzle on the right

In 1735 Euler did NOT draw such a picture

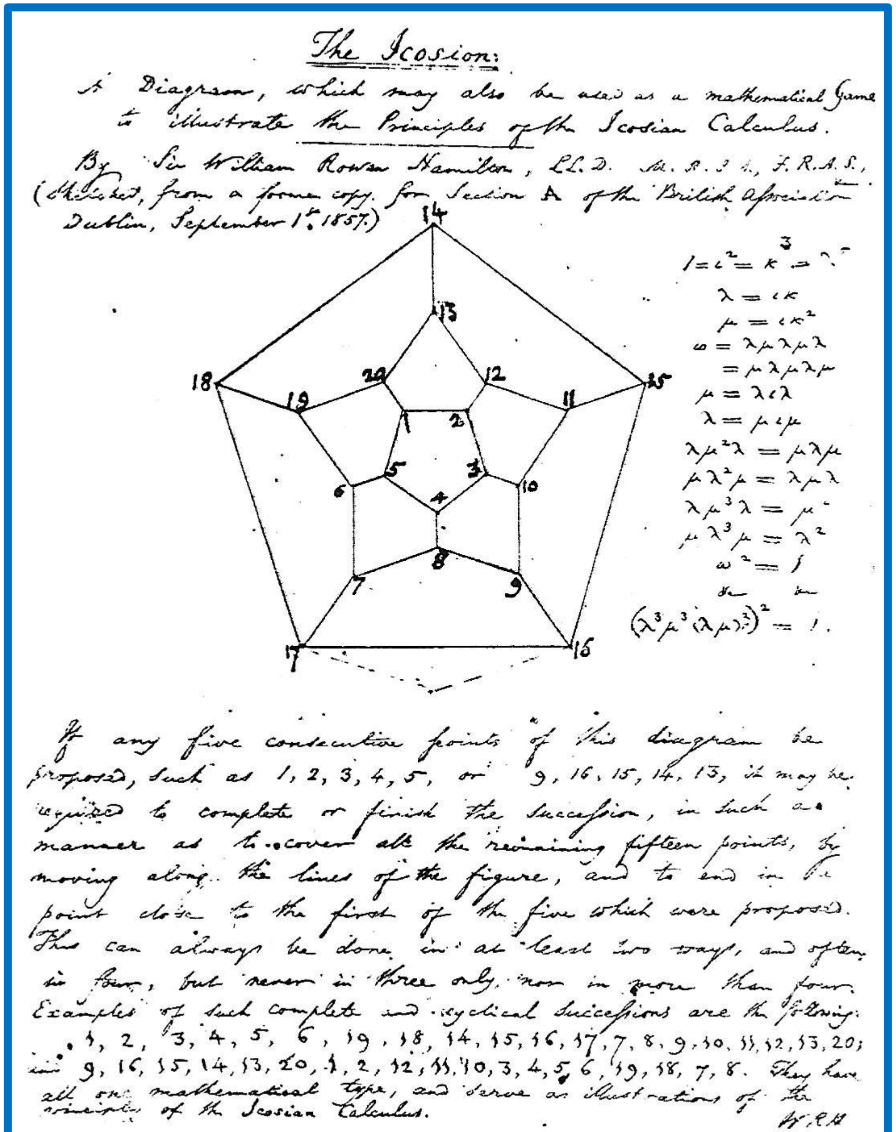
William Rowan Hamilton (1805-1865)

William Rowan Hamilton (1805-1865)



Can we visit each vertex just once and return to our starting point?

Hamilton investigates . . .



The Icosion:

A Diagram, which may also be used as a mathematical Game to illustrate the Principles of the Icosian Calculus.

By Sir William Rowan Hamilton, LL.D. M.R.I.A., F.R.A.S., (Sketched, from a former copy, for Section A of the British Association, Dublin, September 1st 1857.)

$1 = \iota^2 = \kappa^3 = \lambda^5$

$\lambda = \iota\kappa$

$\mu = \iota\kappa^2$

$\omega = \lambda\mu\lambda\mu\lambda$

$= \mu\lambda\mu\lambda\mu$

$\mu = \lambda\iota\lambda$

$\lambda = \mu\iota\mu$

$\lambda\mu^2\lambda = \mu\lambda\mu$

$\mu\lambda^2\mu = \lambda\mu\lambda$

$\lambda\mu^3\lambda = \mu^2$

$\mu\lambda^3\mu = \lambda^2$

$\omega^2 = 1$

&c &c

$(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1.$

If any five consecutive points of this diagram be proposed, such as 1, 2, 3, 4, 5, or 9, 16, 15, 14, 13, it may be required to complete or finish the succession, in such a manner as to cover all the remaining fifteen points, by moving along the lines of the figure, and to end in the point close to the first of the five which were proposed. This can always be done in at least two ways, and often in four, but never in three only, nor in more than four. Examples of such complete and cyclical successions are the following: 1, 2, 3, 4, 5, 6, 19, 18, 14, 15, 16, 17, 7, 8, 9, 10, 11, 12, 13, 20; and 9, 16, 15, 14, 13, 20, 1, 2, 12, 11, 10, 3, 4, 5, 6, 19, 18, 7, 8. They have all one mathematical type, and serve as illustrations of the principles of the Icosian Calculus.

W.R.H.

icosian calculus:

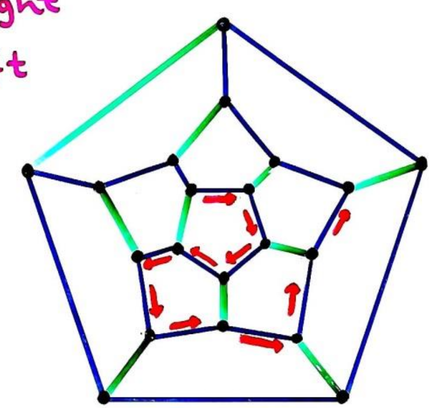
$i^2 = k^3 = l^5 = 1$, where $l = ik$

Let $m = ik^2 = lk$: then

$\underline{l^3 m^3 l m l m l^3 m^3 l m l m = 1}$

l = right

m = left



Hamilton's 'quaternions':

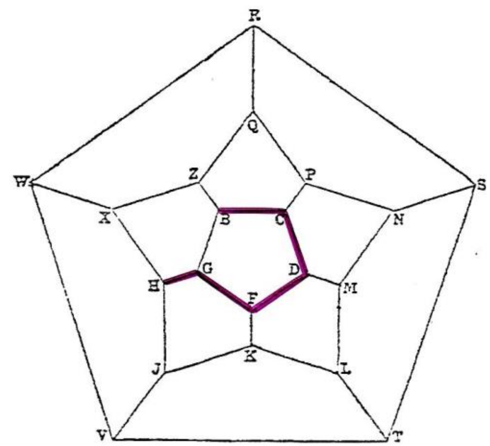
$i^2 = j^2 = k^2 = -1$, $ijk = -1$.

So $ij = k$, $ji = -k$. etc.

Hamilton's icosian game (1859)

THE ICOSIAN GAME.

Entered at Stationers' Hall.



Registered agreeably to Act V. & VI. Vic. cap. 100.

In this new Game (invented by Sir William Rowan Hamilton, LL.D., &c., of Dublin, and by him named *Icosian*, from a Greek word signifying "twenty") a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn in such a manner as always to proceed *along the lines* of the figure, and also to fulfil certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in the hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

LONDON:

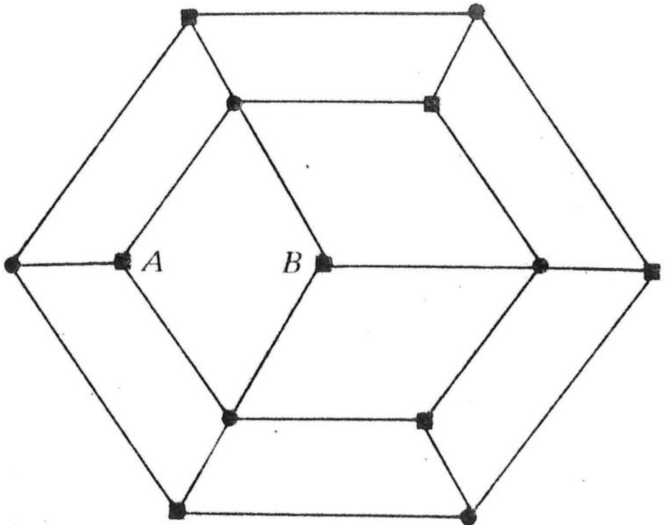
PUBLISHED AND SOLD WHOLESALE BY JOHN JAQUES AND SON, 102 HATTON GARDEN;

AND TO BE HAD AT MOST OF THE LEADING FANCY REPOSITORIES THROUGHOUT THE KINGDOM.

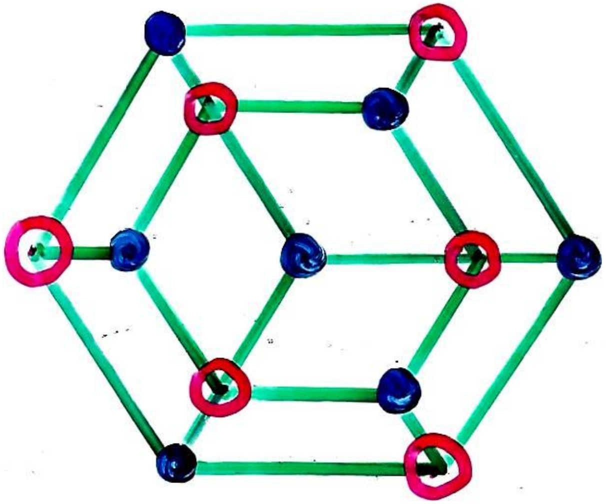
Marketing the icosian game

Hamilton sold the icosian game to a games manufacturer for £25 – a wise move, as it didn't sell.

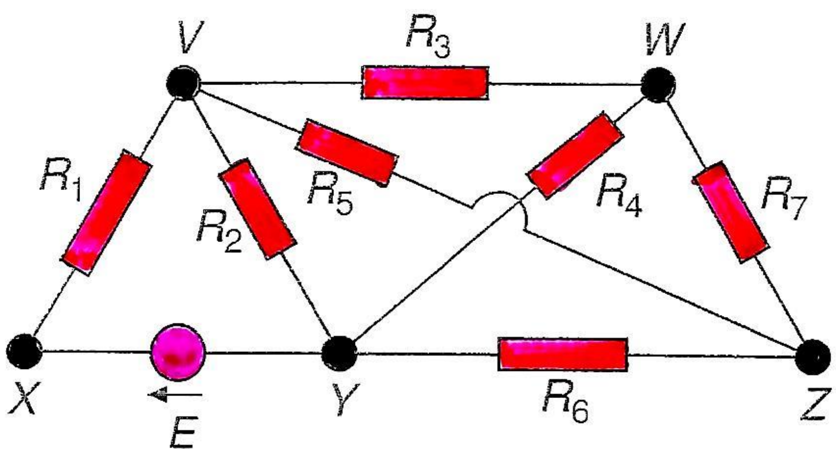
Kirkman's 'cell of a bee' (1856)



Kirkman investigated closed paths on 'polyedra'.
But if we cut in two the cell of a bee (giving the picture above), is there such a path?

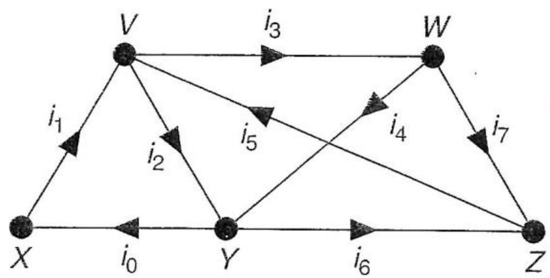


Kirchhoff's electrical networks (1845/7)



Problem: Find all the currents and voltages

(using Kirchhoff's laws)

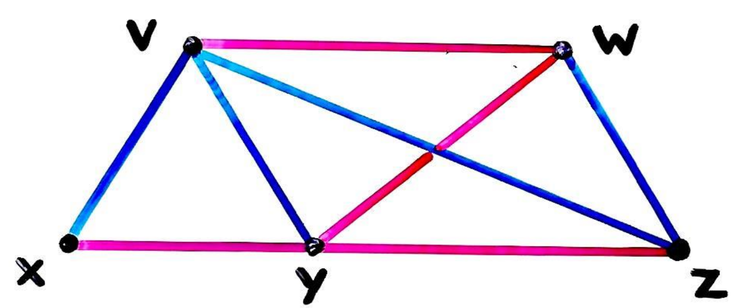


assign directions to currents

cycle VXYV: $i_1R_1 + i_2R_2 = E$

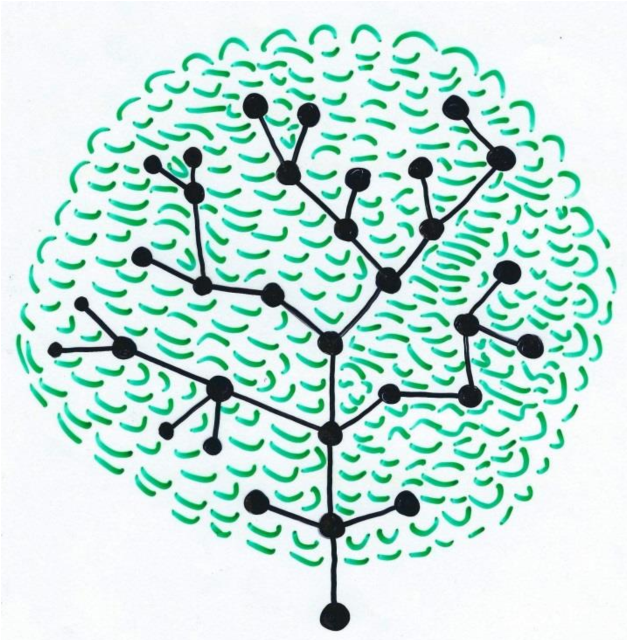
cycle VWYV: $i_3R_3 + i_4R_4 - i_2R_2 = 0$

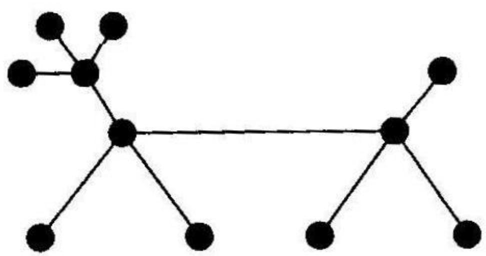
cycle VWYXV: $i_1R_1 + i_3R_3 + i_4R_4 = E$

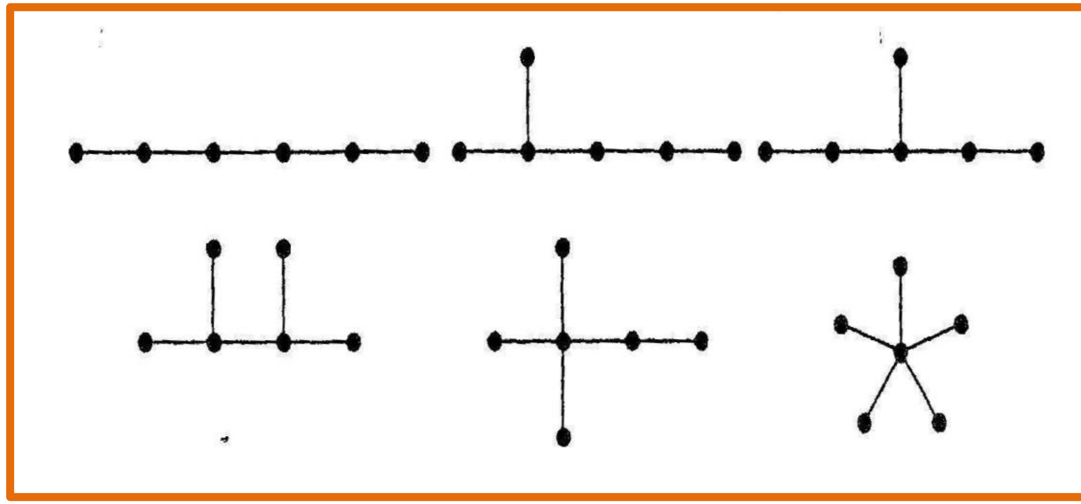


Fundamental set of cycles for a spanning tree

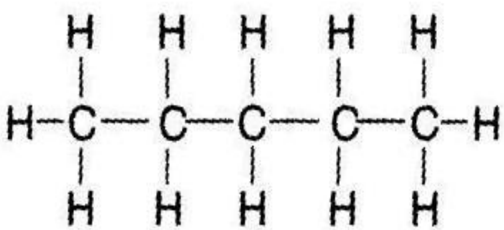
Counting trees

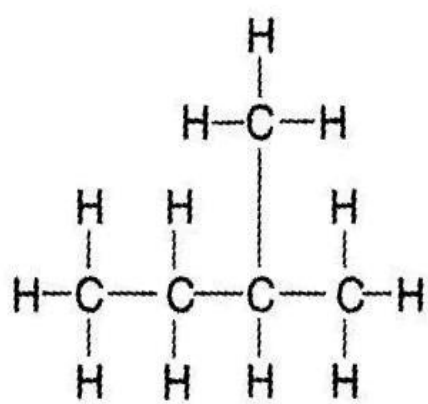






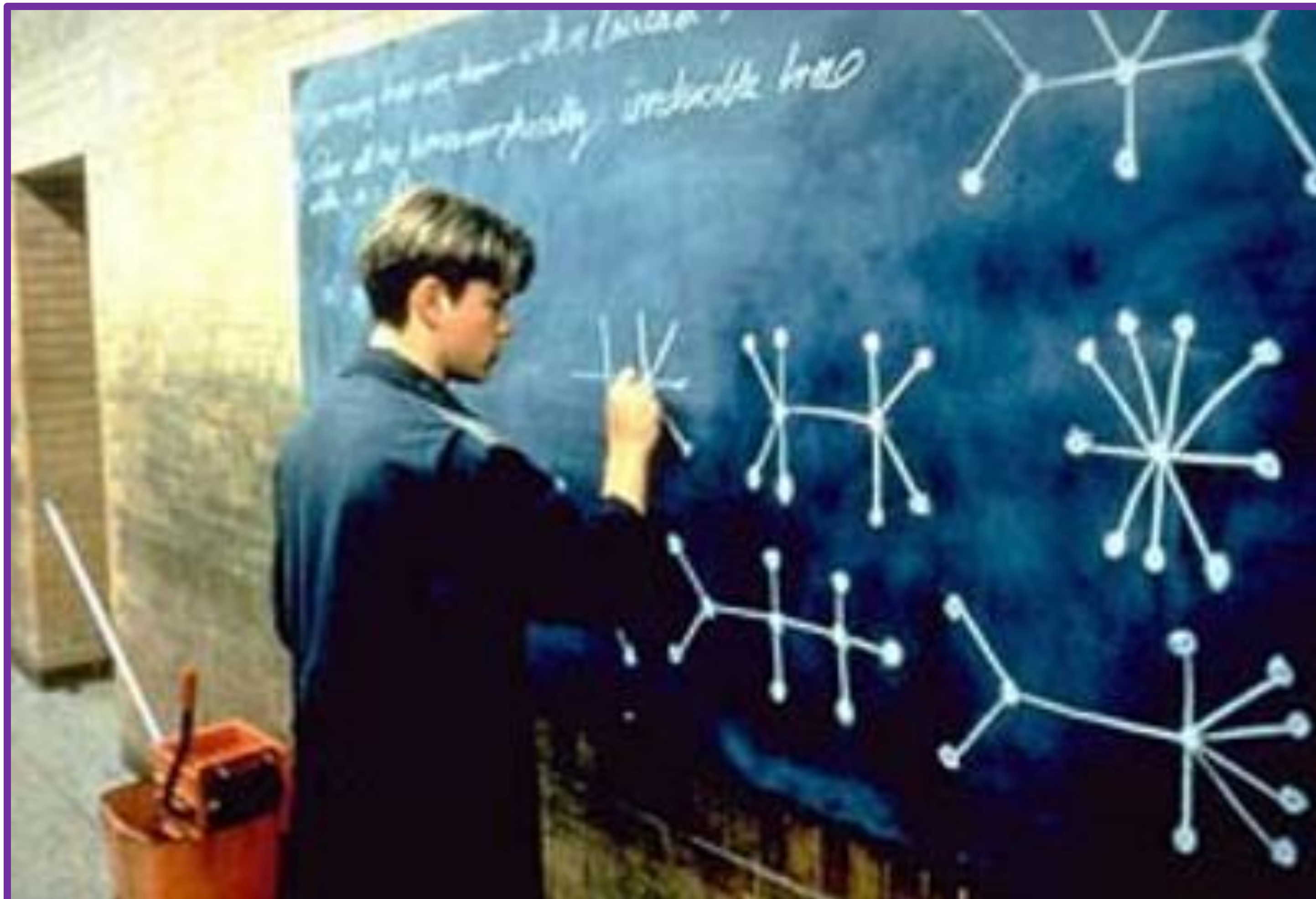
The six trees with 6 vertices:
how many trees have 100 vertices?





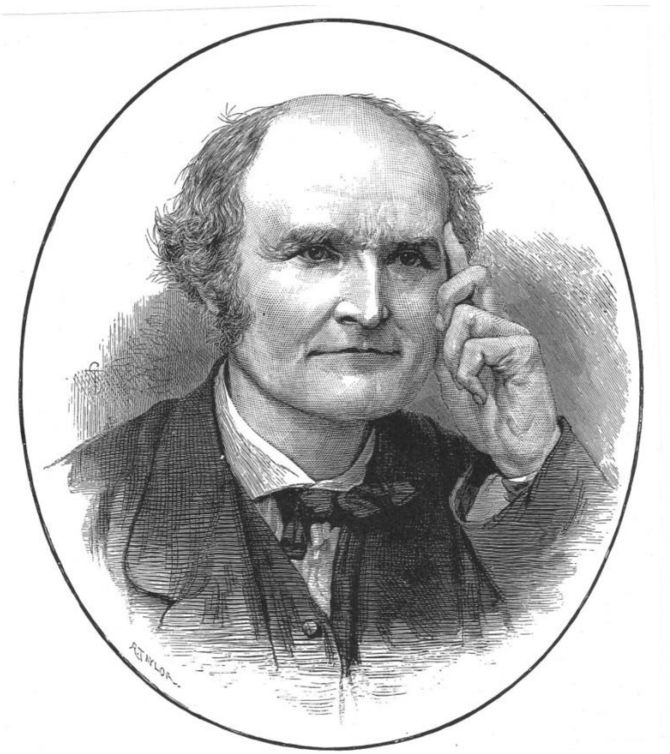
Alkanes C_nH_{2n+2} have a tree structure: how many have n carbon atoms?

Good Will Hunting



Draw all the homeomorphically irreducible trees with 10 vertices

Cayley's trees (1857/9)

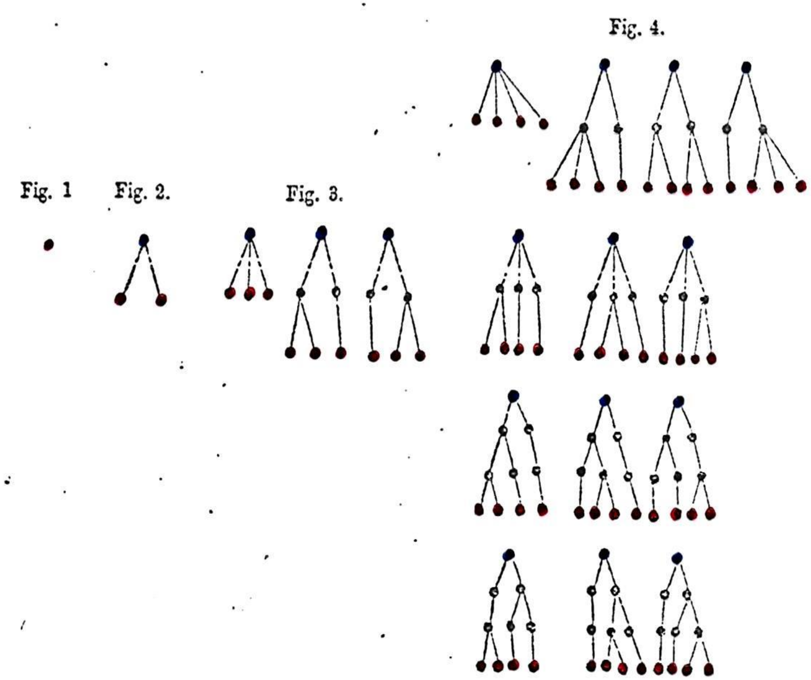


Arthur Cayley

ON THE ANALYTICAL FORMS CALLED TREES. SECOND PART.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 374—378. Continuation of 203.]

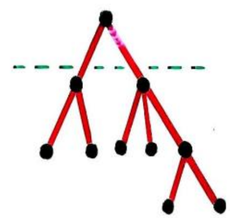
THE following class of "trees" presented itself to me in some researches relating to functional symbols; viz., attending only to the terminal knots, the trees with one knot, two knots, three knots, and four knots respectively are shown in the figures 1, 2, 3 and 4:



and similarly for any number of knots. The trees with four knots are formed first from those of one knot by attaching thereto in every possible way (one way only)

Cayley's 1881 paper

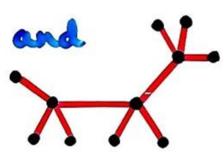
To count rooted trees, Cayley used a generating function $1 + A_1 x + A_2 x^2 + \cdots$ where A_n = number of rooted trees with n branches:



he obtained the equation

$$1 + A_1 x + A_2 x^2 + \cdots = (1-x)^{-1} \cdot (1-x^2)^{-A_1} \cdot (1-x^3)^{-A_2} \cdots$$

To count unrooted trees, he started from the 'centre' of the tree and worked outwards.



n:	1	2	3	4	5	6	7	8	9	10	...
rooted:	1	2	4	9	20	48	115	286	719	1842	...
unrooted:	1	1	2	3	6	11	23	47	106	235	...

1881

On the Analytical Forms called Trees.

By Professor Cayley.

In a tree of N knots, selecting any knot at pleasure as a root, the tree may be regarded as springing from this root, and it is then called a root-tree. The same tree thus presents itself in various forms as a root-tree; and if we consider the different root-trees with N knots, these are not all of them distinct trees. We have thus the two questions, to find the number of root-trees with N knots: and, to find the number of distinct trees with N knots.

I have in my paper "On the Theory of the Analytical Forms called Trees," *Phil. Mag.*, t. 13 (1857), pp. 172–176, given the solution of the first question; viz. if ϕ_N denotes the number of the root-trees with N knots, then the successive numbers ϕ_1, ϕ_2, ϕ_3, etc., are given by the formula

$$\phi_1 + x\phi_2 + x^2\phi_3 + \ldots = (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3}\ldots$$

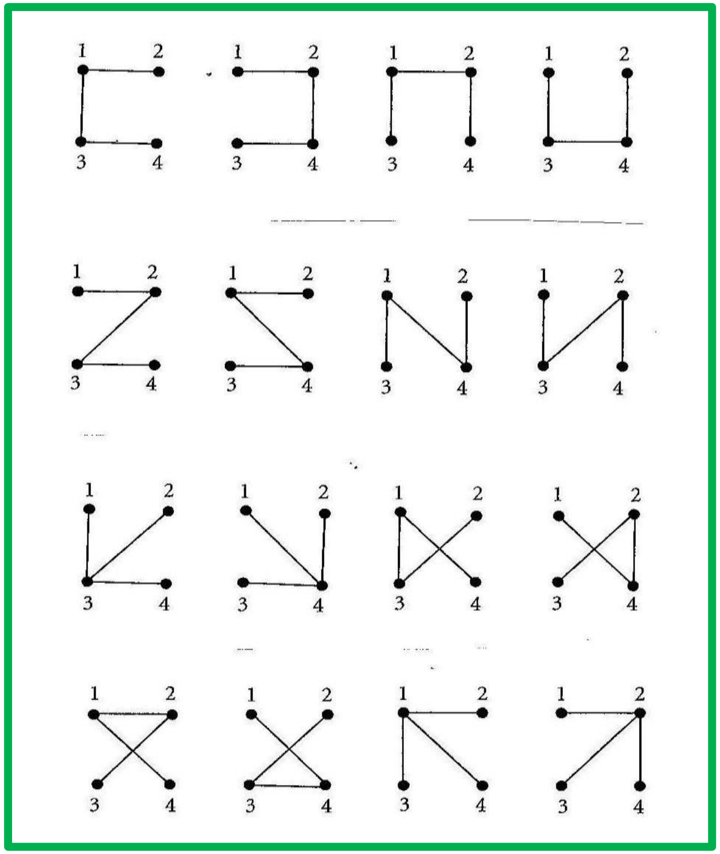
viz. we thus find

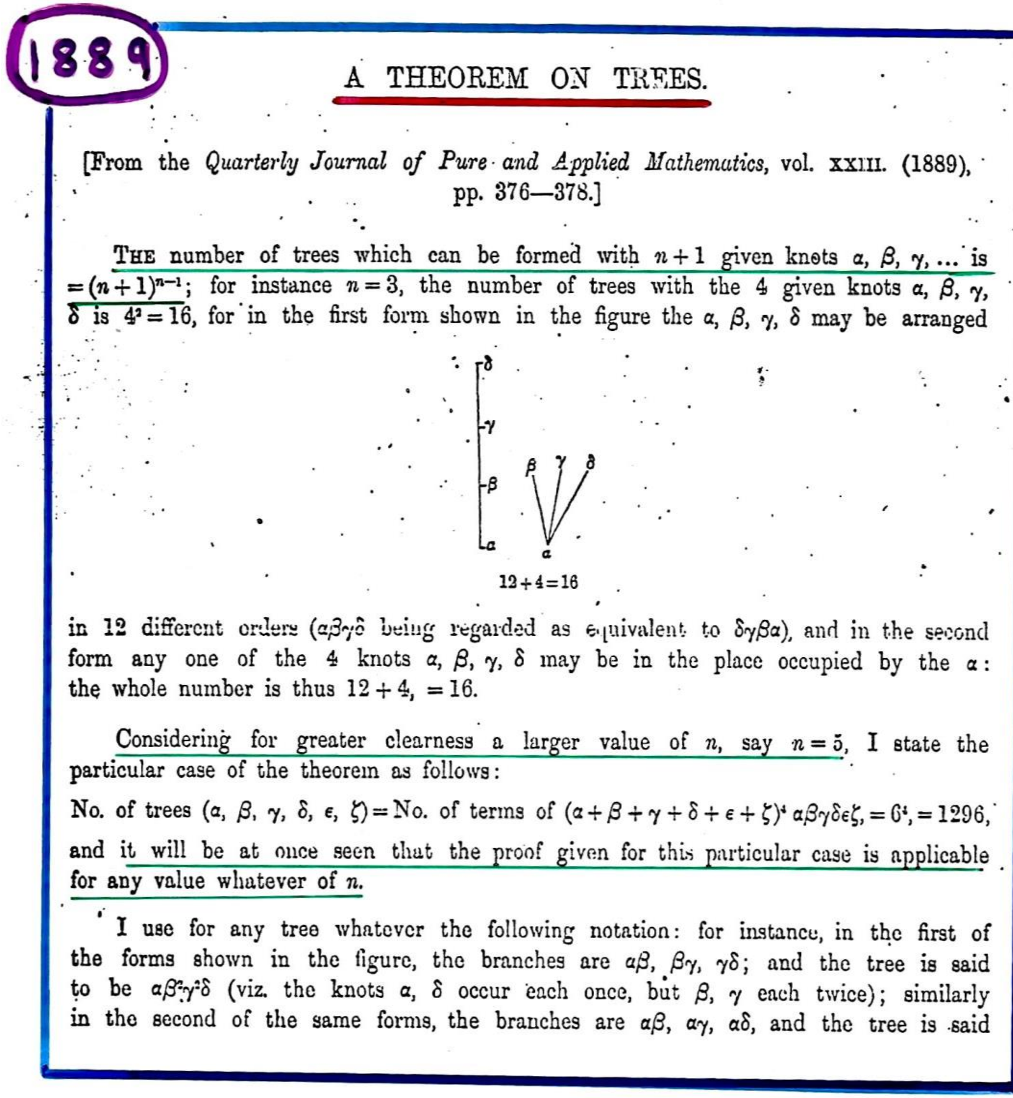
suffix of ϕ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi =$	1	1	2	4	9	20	48	115	286	719	1842	4766	12486

And I have, in the paper "On the Analytical Forms called Trees, with Applications to the Theory of Chemical Combinations," *Brit. Assoc. Report*, 1875, pp. 257–305, also shown how by the consideration of the centre or bicentre "of length" we can obtain formulae for the number of central and bicentral trees, that is for the number of distinct trees, with N knots: the numerical result obtained for the total number of distinct trees with N knots is given as follows:

No. of Knots	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of Central Trees	1	0	1	1	2	3	7	12	27	55	127	284	682
" Bicentral "	0	1	0	1	1	3	4	11	20	51	108	267	619
Total	1	1	1	2	3	6	11	23	47	106	235	551	1301

Counting labelled trees





1889

A THEOREM ON TREES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII. (1889), pp. 376—378.]

THE number of trees which can be formed with $n+1$ given knots $\alpha, \beta, \gamma, \ldots$ is $=(n+1)^{n-1}$; for instance $n=3$, the number of trees with the 4 given knots $\alpha, \beta, \gamma, \delta$ is $4^2=16$, for in the first form shown in the figure the $\alpha, \beta, \gamma, \delta$ may be arranged

δ γ β α β γ δ α

$12+4=16$

in 12 different orders ($\alpha\beta\gamma\delta$ being regarded as equivalent to $\delta\gamma\beta\alpha$), and in the second form any one of the 4 knots $\alpha, \beta, \gamma, \delta$ may be in the place occupied by the α: the whole number is thus $12+4$, $=16$.

Considering for greater clearness a larger value of n, say $n=5$, I state the particular case of the theorem as follows:

No. of trees $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$ = No. of terms of $(\alpha+\beta+\gamma+\delta+\epsilon+\zeta)^4\alpha\beta\gamma\delta\epsilon\zeta$, $=6^4$, $=1296$,

and it will be at once seen that the proof given for this particular case is applicable for any value whatever of n.

I use for any tree whatever the following notation: for instance, in the first of the forms shown in the figure, the branches are $\alpha\beta$, $\beta\gamma$, $\gamma\delta$; and the tree is said to be $\alpha\beta^2\gamma^2\delta$ (viz. the knots α, δ occur each once, but β, γ each twice); similarly in the second of the same forms, the branches are $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, and the tree is said

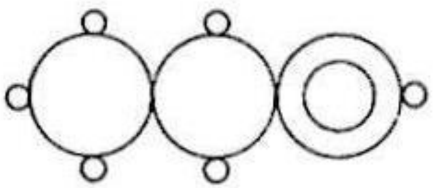
Arthur Cayley, 1889:

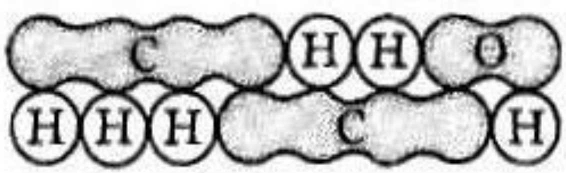
The number of n-vertex labelled trees is n^{n-2}.

Chemical diagrams – 'graphic formulae'

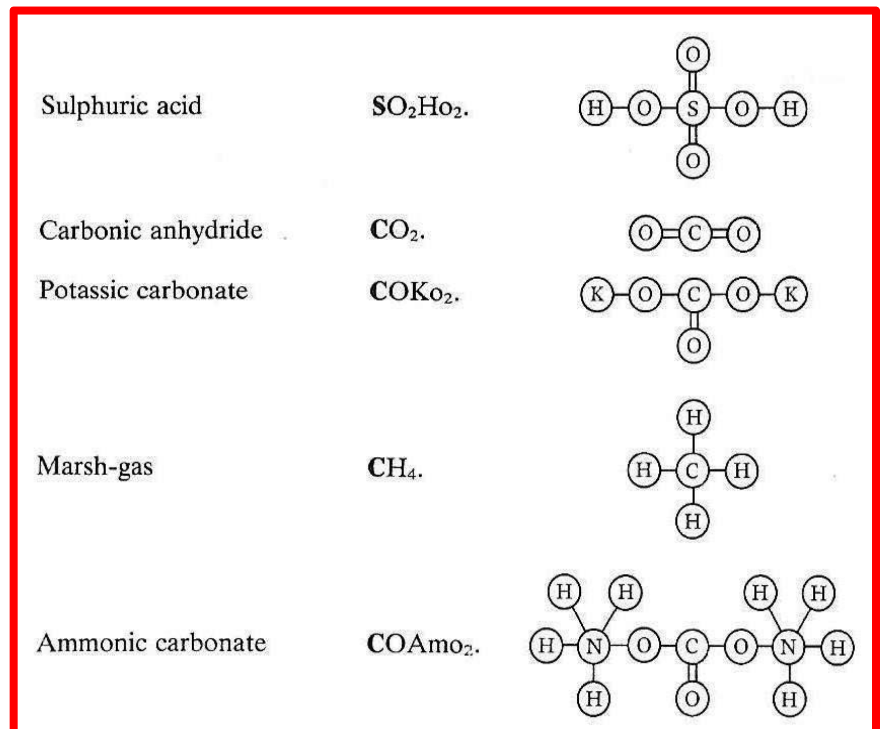
By 1850 it was known that elements combined in fixed proportions to make compounds– formulas such as C_2H_5OH were known – but how did elements combine?

Answer: VALENCY





In 1864 Alexander Crum Brown introduced his 'graphic formulae', which then appeared in a popular textbook by Edward Frankland.



Sulphuric acid	SO_2Ho_2.	
Carbonic anhydride	CO_2.	
Potassic carbonate	$COKo_2$.	
Marsh-gas	CH_4.	
Ammonic carbonate	$COAmo_2$.	

J. J. Sylvester (Nature, 1878)

. . . I hardly ever take up Dr. Frankland's exceedingly valuable *Notes for Chemical Students*, which are drawn up exclusively on the basis of Kekule's exquisite concept of *valence*, without deriving suggestions for new researches in the theory of algebraical forms . . .

The analogy is between atoms and binary *quantics exclusively. I compare every binary quantic with a chemical atom. The number of factors ... in a binary quantic is the analogue of the number of* bonds, *or the* valence, *... of a chemical atom...*

An invariant of a system of binary quantics of various degrees is the analogue of a chemical substance composed of atoms of corresponding valences... *A co-variant is the analogue of an (organic or inorganic) compound radical...*

radical. Every invariant and covariant thus becomes expressible by a graph *precisely identical with a Kekuléan diagram or chemicograph...*

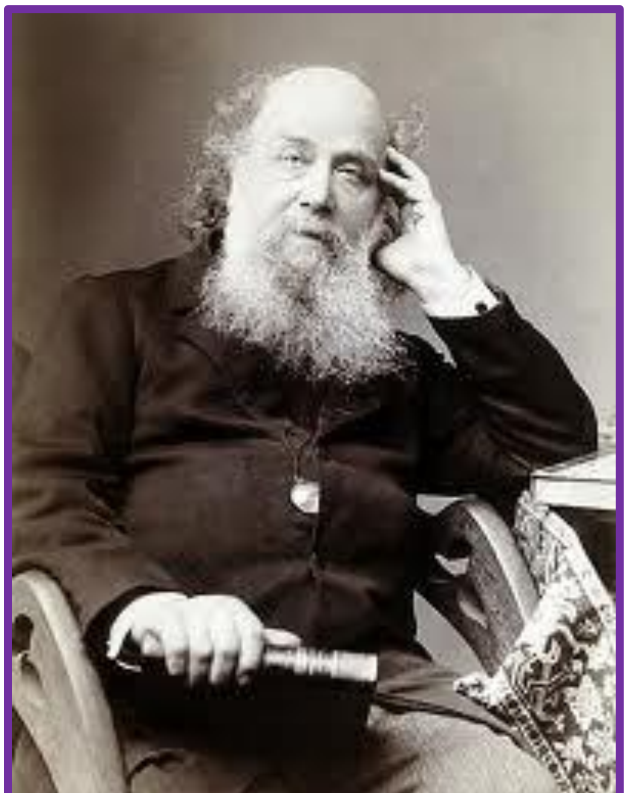
Chemistry and algebra

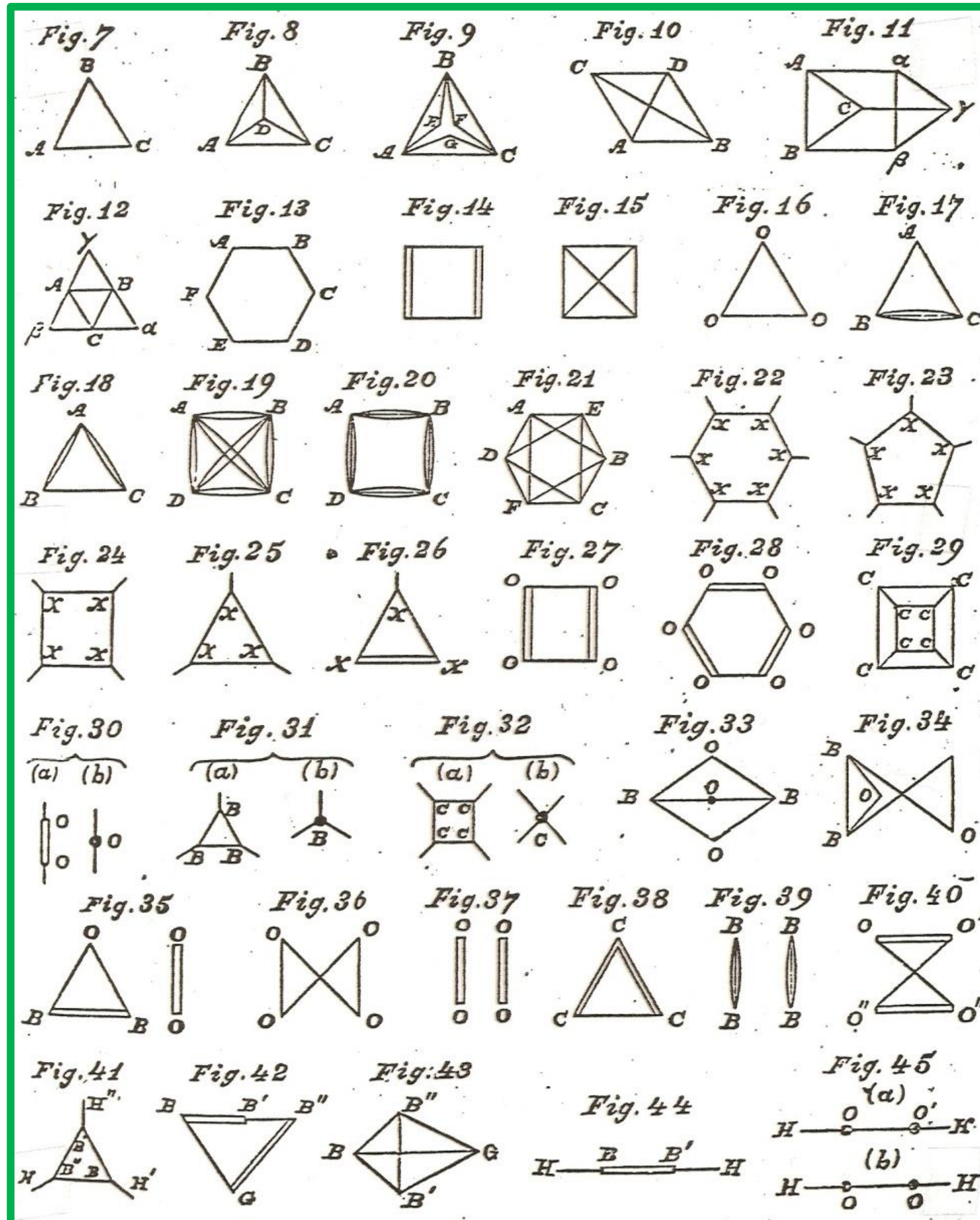
***American Journal of Mathematics*, 1878**

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson.

Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other.

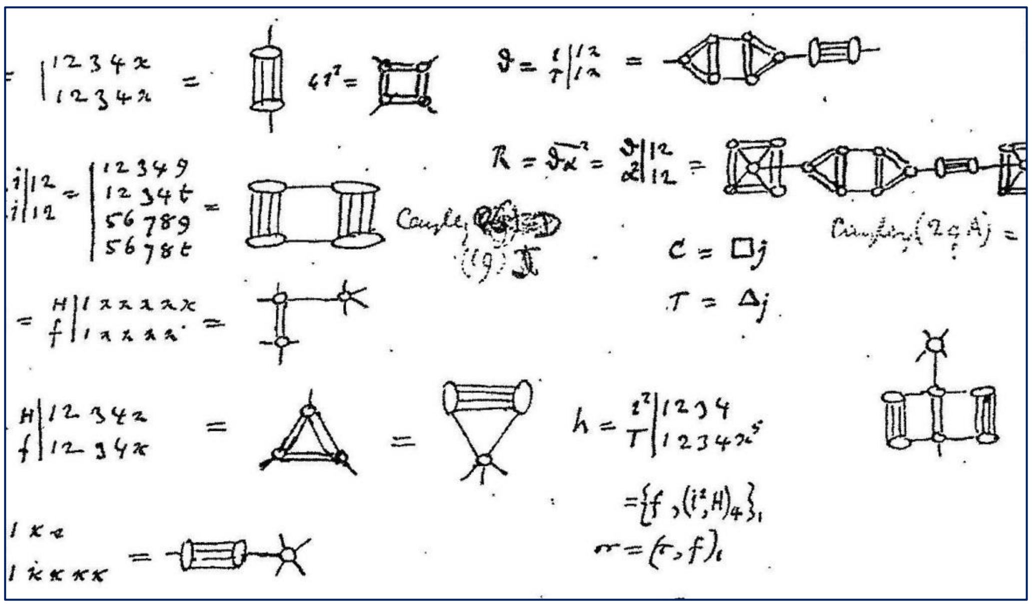
Sylvester's chemical trees (1878)





J. J. Sylvester and W. K. Clifford

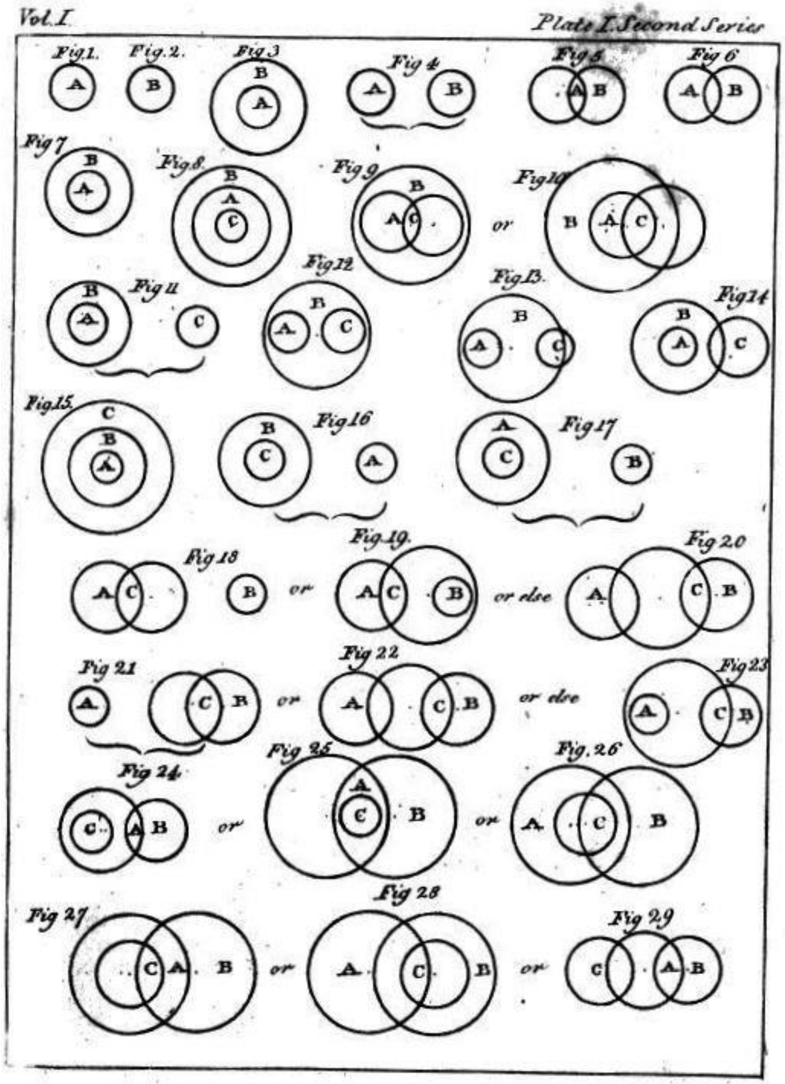
atom ↔ binary quantic ($ax^3 + 3bx^2y + 3cxy^2 + dy^3$)
number of bonds ↔ number of factors
chemical substance ↔ invariant (functions of a, b, c, . . .)
invariant/covariant ↔ graph



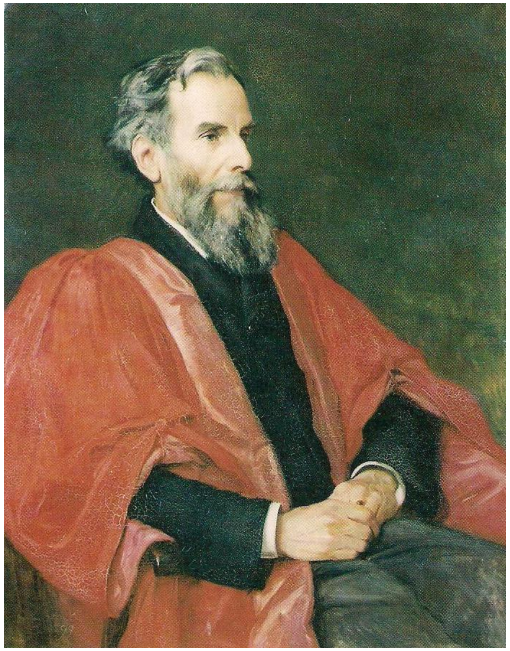
Cayley and Sylvester's work on invariant theory was eventually supplanted by the more powerful methods of Gordan and Hilbert – the 'finite basis theorem'.

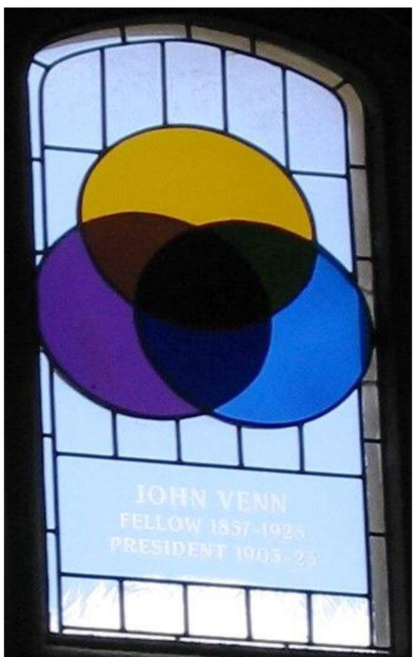
Euler diagrams (1761)

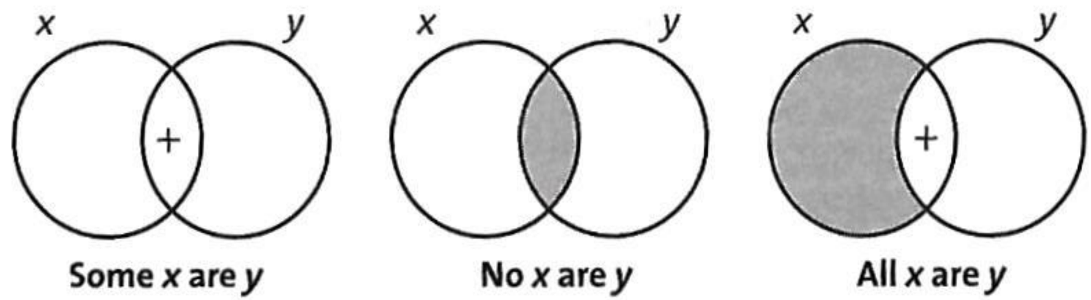

Briefe
an eine
deutsche Prinzessinn
über verschiedene Gegenstände
aus der Physik und Philosophie.
Aus dem Französischen übersetzt.
Zweyter Theil.
Leipzig,
bey Johann Friedrich Junius,
1769.

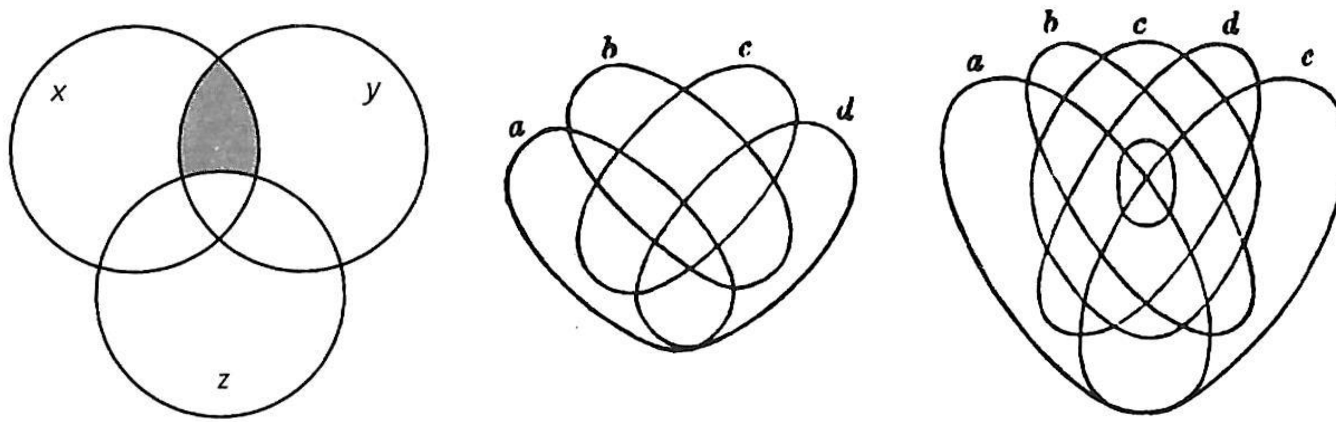


Venn diagrams (1881)



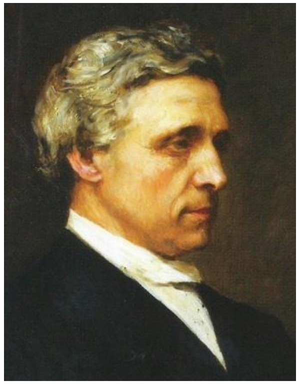


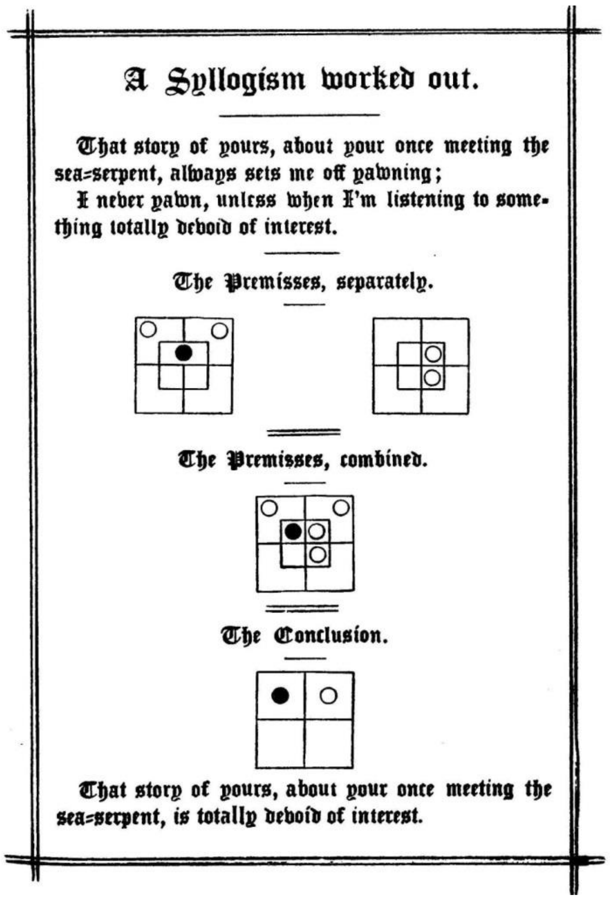




Venn diagrams for 2, 3, 4, 5 sets

Lewis Carroll's diagrams





A Syllogism worked out.

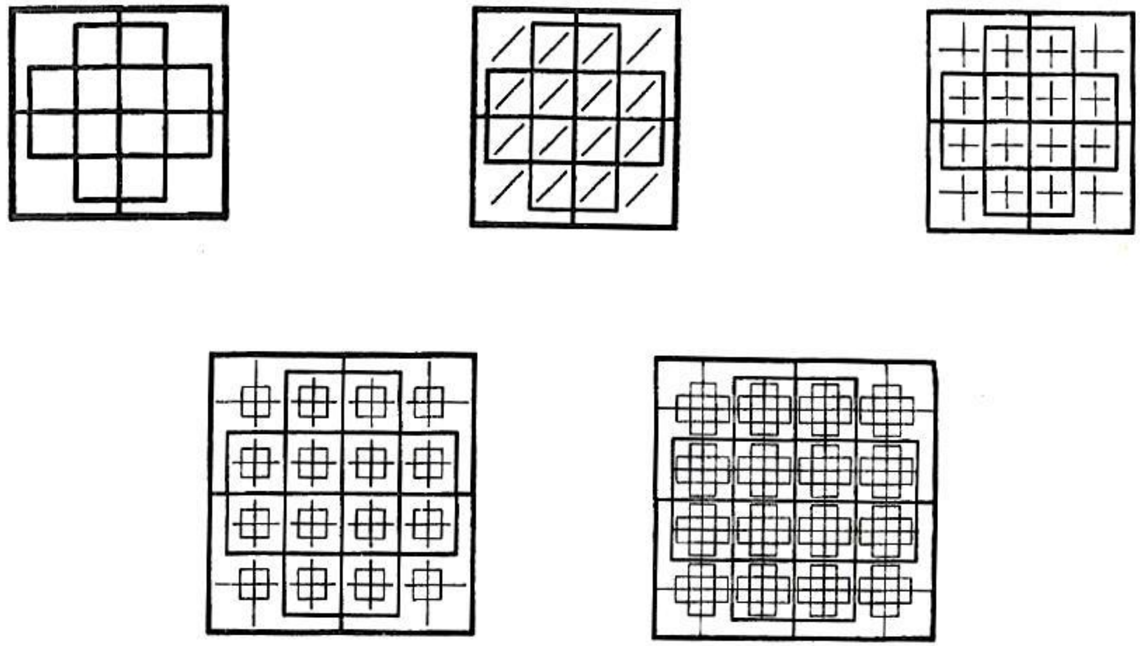
That story of yours, about your once meeting the sea-serpent, always sets me off yawning;
I never yawn, unless when I'm listening to something totally devoid of interest.

The Premisses, separately.

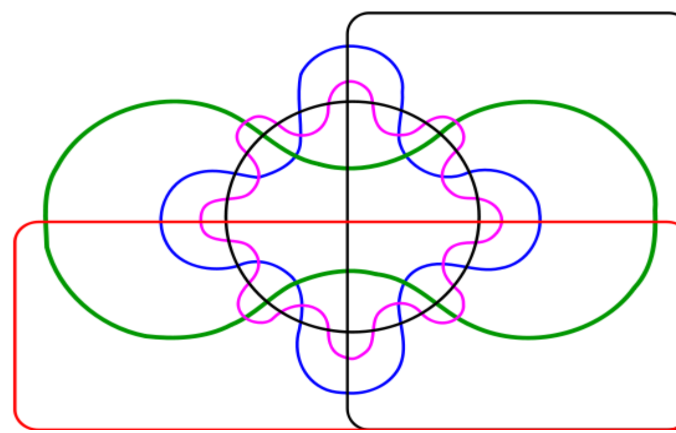
The Premisses, combined.

The Conclusion.

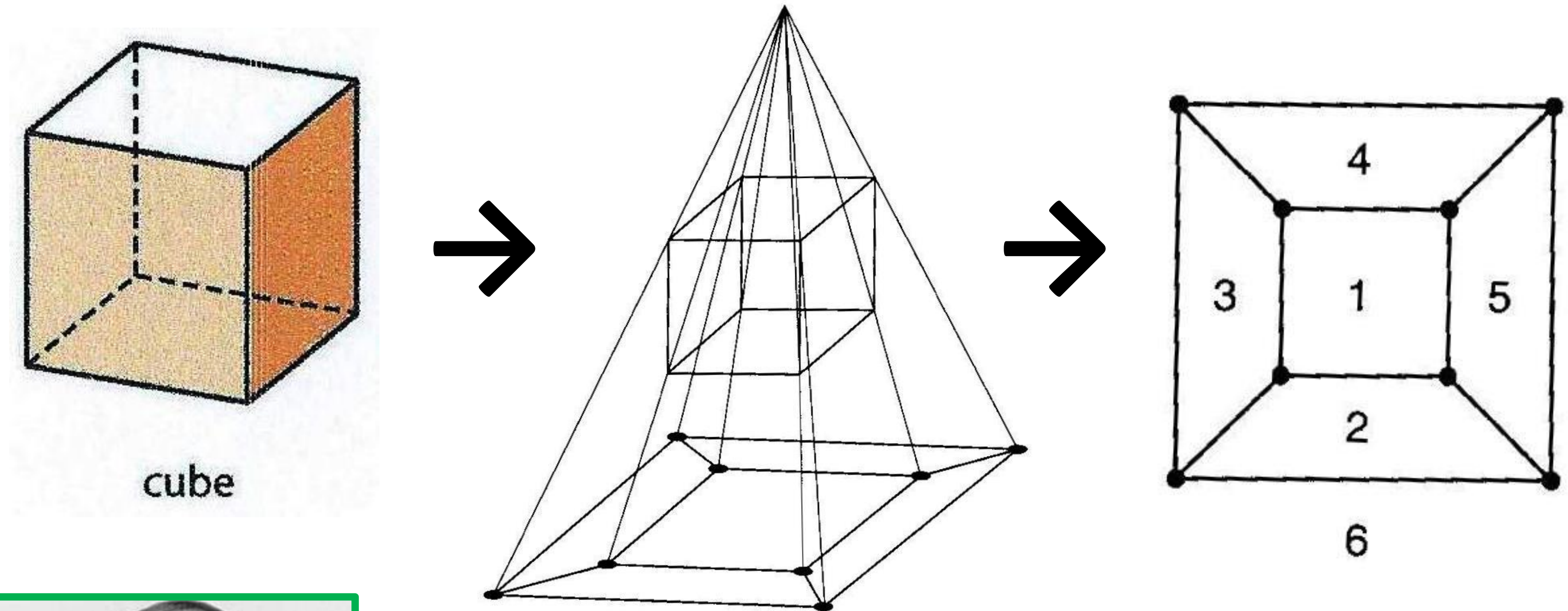
That story of yours, about your once meeting the sea-serpent, is totally devoid of interest.



A. W. F. Edwards: *Cogwheels of the Mind*



Euler's formula for plane graphs

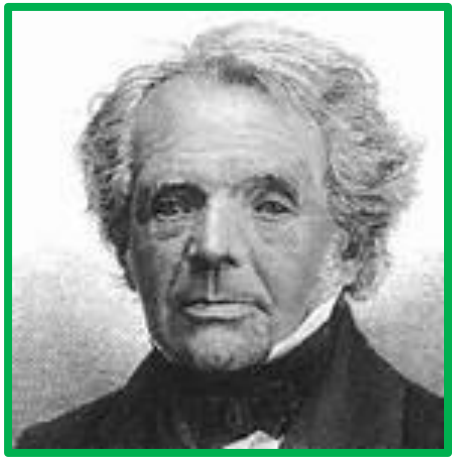


(number of faces) + (number of vertices)
= (number of edges) + 2

6 + 8 = 12 + 2

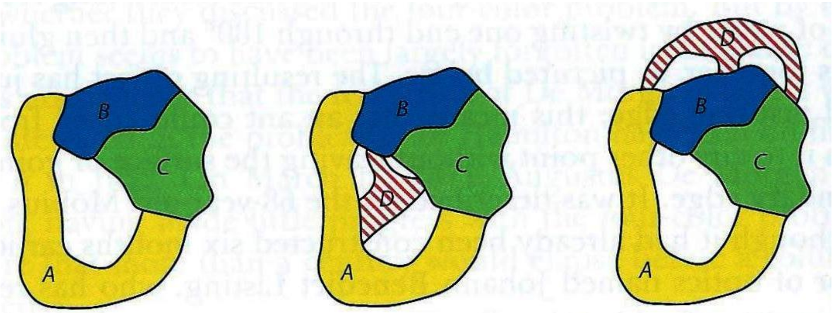
Augustin-Louis Cauchy

Möbius's problem (c.1840)

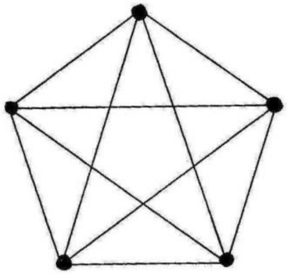


A king lay on his death-bed:
'My five sons, divide my land among you, so that each part has a border with each of the others.'

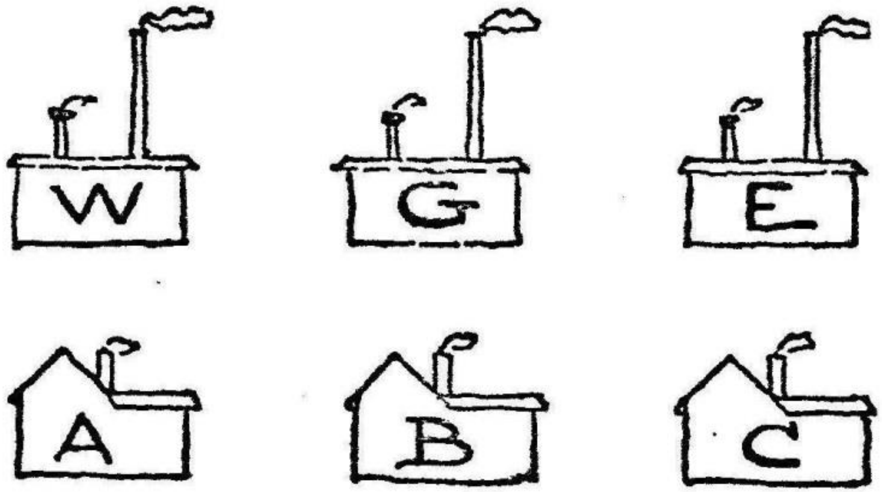
Möbius's problem has no solution: five neighbouring regions cannot exist

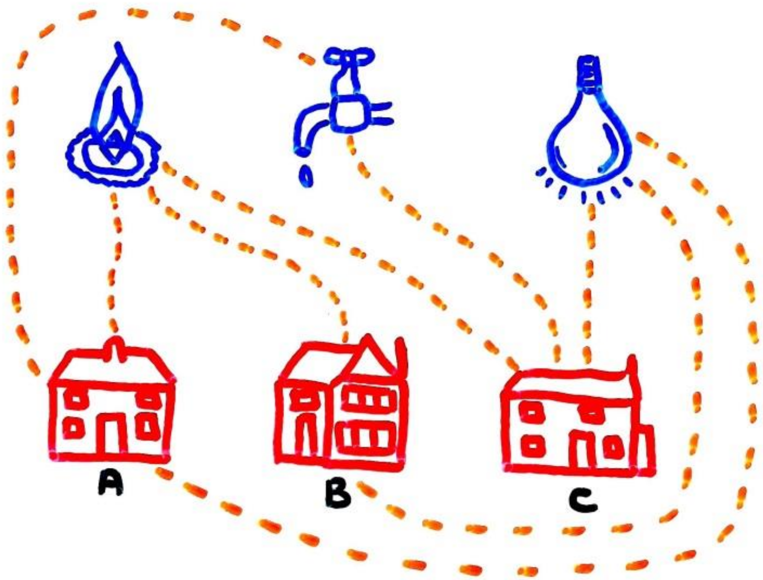


Dual form:
Can you join five towns by non-crossing roads?
no – so K_5 is non-planar



The utilities problem (Sam Loyd, 1900)

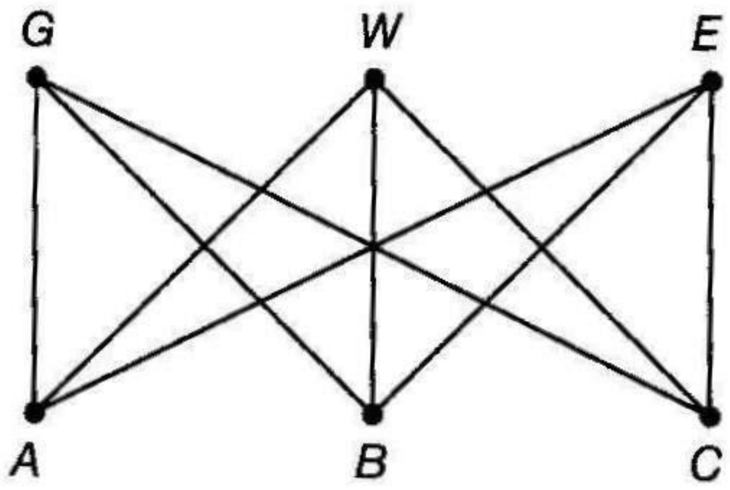




Can we connect the three houses A, B, C to the three utilities gas, water, electricity without any of the connections crossing?

(Here, house B is not joined to water)

Solving the utilities problem



Is this graph $K_{3,3}$ planar?

Look at the 6-cycle A-G-B-W-C-E-A, and try to add the connections A-W, G-C, and E-B . . .