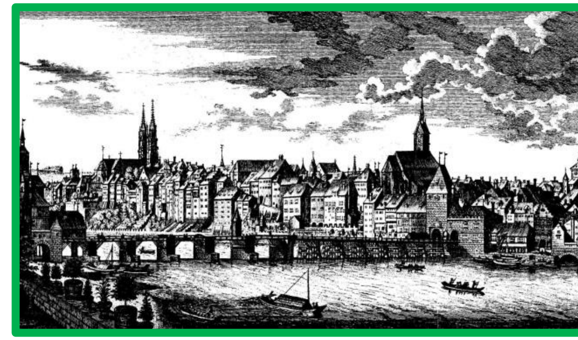
3. Euler’s Combinatorics

Robin Wilson

Some of Euler's interests

Theory of numbers
Geometry of a triangle
Musical harmony
Infinite series
Logarithms
Calculus
Mechanics
Complex numbers
Optics
Astronomy
Motion of the moon
Wave motion
Stability of sailing ships . . .

Summary of Euler's life



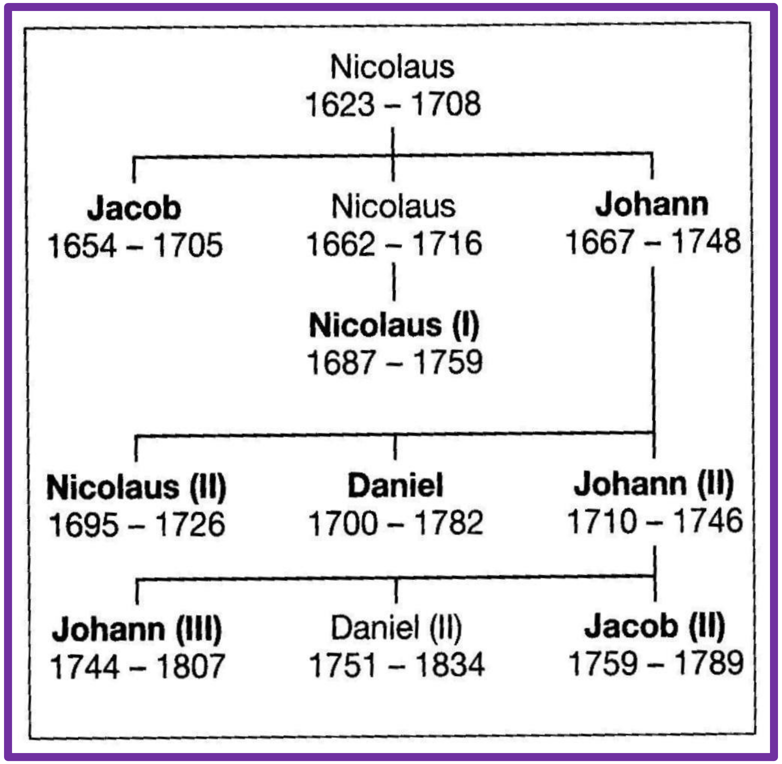
1707: Born in Basel (15 April)
1721: University of Basel

1727: St Petersburg Academy
1733: Chair of Mathematics

1741: Berlin Academy of Sciences

1766: returned to St Petersburg
1783: died in St Petersburg

The Bernoulli family



Euler's combinatorics

1735:	Königsberg bridges problem
1741-68:	Partitions
1750s:	Polyhedron formula
1751:	Dividing polygons
1753, 1779:	Derangements
1759:	Knight's-tour problem
1771:	Josephus problem
[1776:	Binomial coefficients]
[1776-82:	Magic squares and Latin squares]

The 1730s in St Petersburg

1732: $2^{32} + 1$ is divisible by 641

1735: $1 + {}^1/_4 + {}^1/_9 + {}^1/_{16} + \ldots = \pi^2/6$

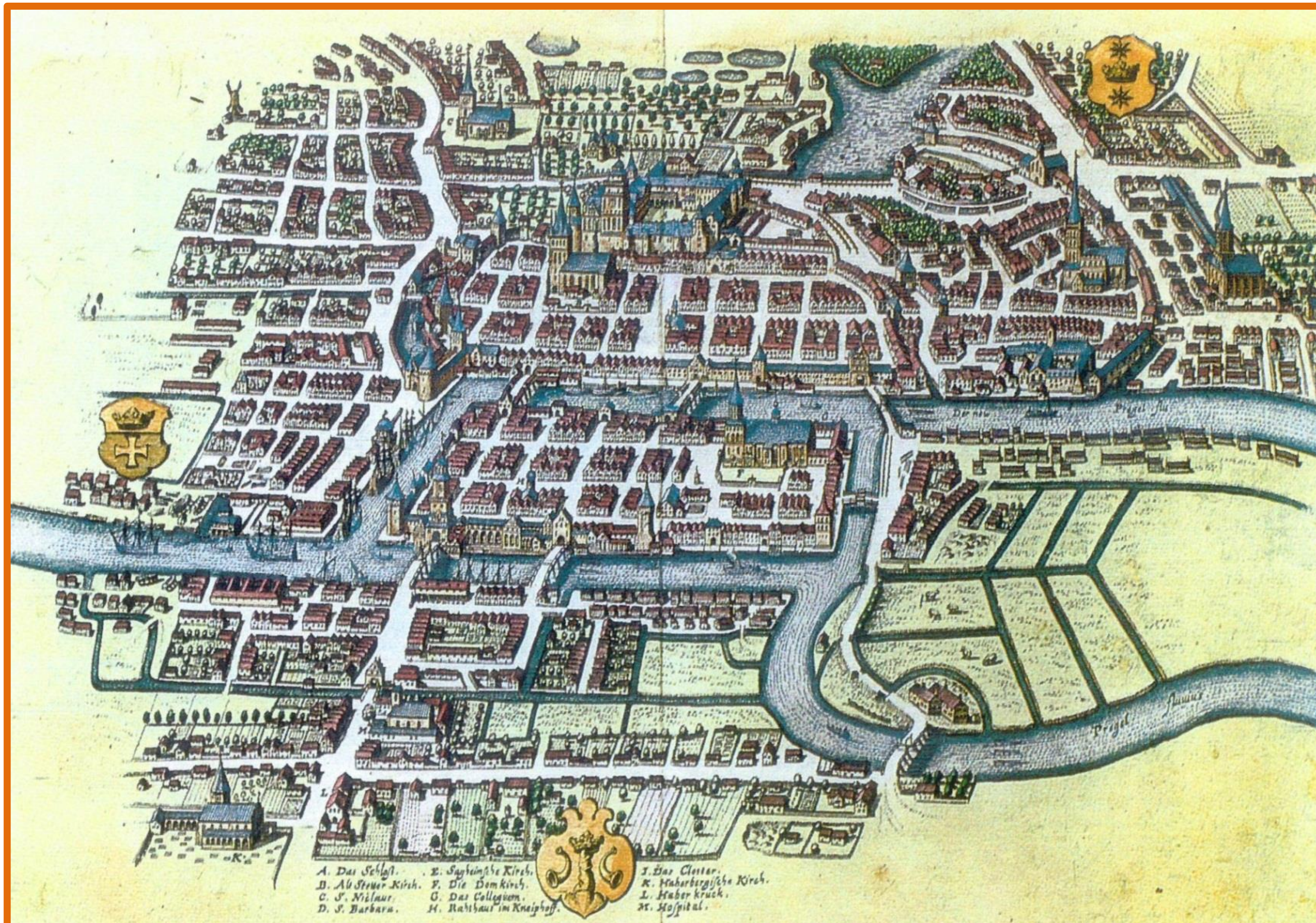
1735: Königsberg bridges

1736: *Mechanica*

1737: e is irrational

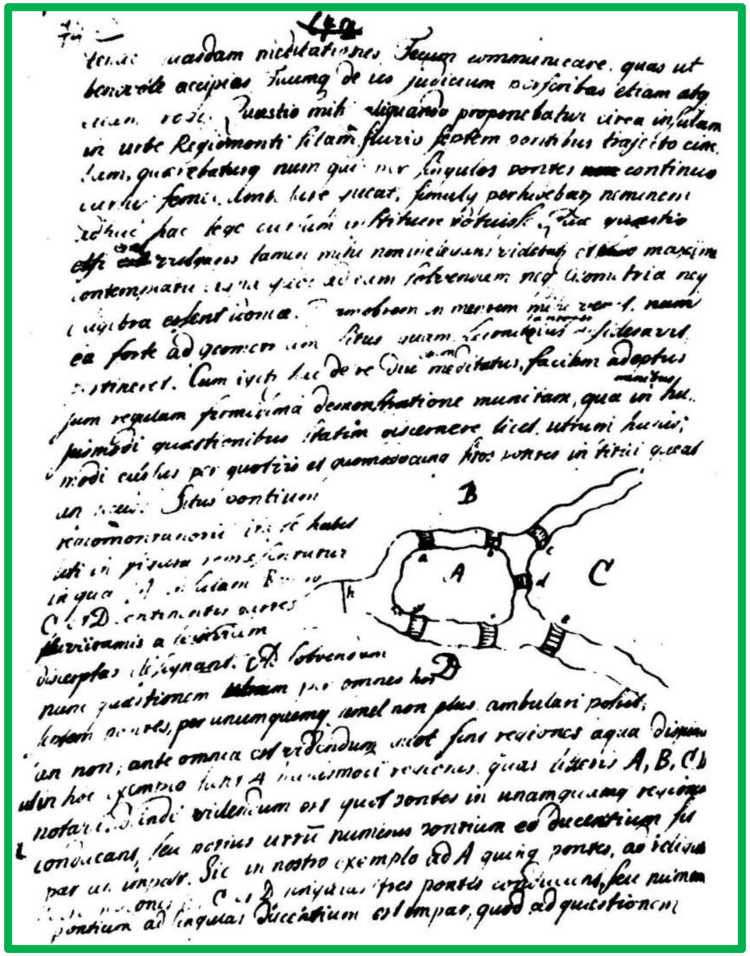
- Calculus of variations
- Analytic number theory
- Continued fractions
- Musical theory of harmony
- Cartography

Königsberg bridges problem (1735)



Can you cross each of the seven bridges exactly once?

Euler and the Königsberg bridges



This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it.

In view of this, it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for.

And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not ...

Letter dated 13 March 1736 to Giovanni Marinoni, Court Astronomer to Kaiser Leopold in Vienna,

Solving the Königsberg bridges problem

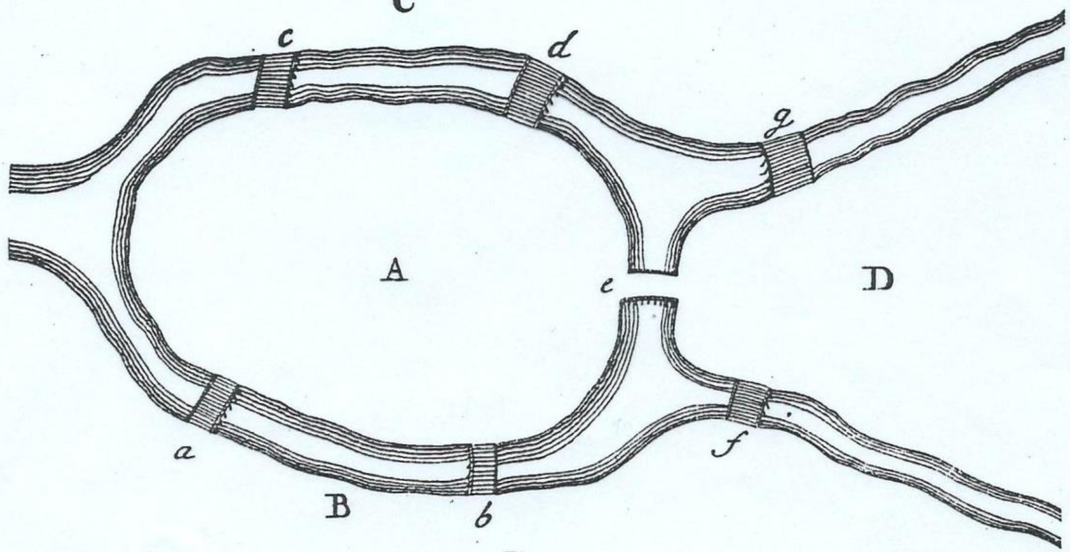


Fig. 1.

		16
*A**,	8	4
*B**,	4	2
*C**,	4	2
D ,	3	2
E ,	5	3
*F**,	6	3
		16

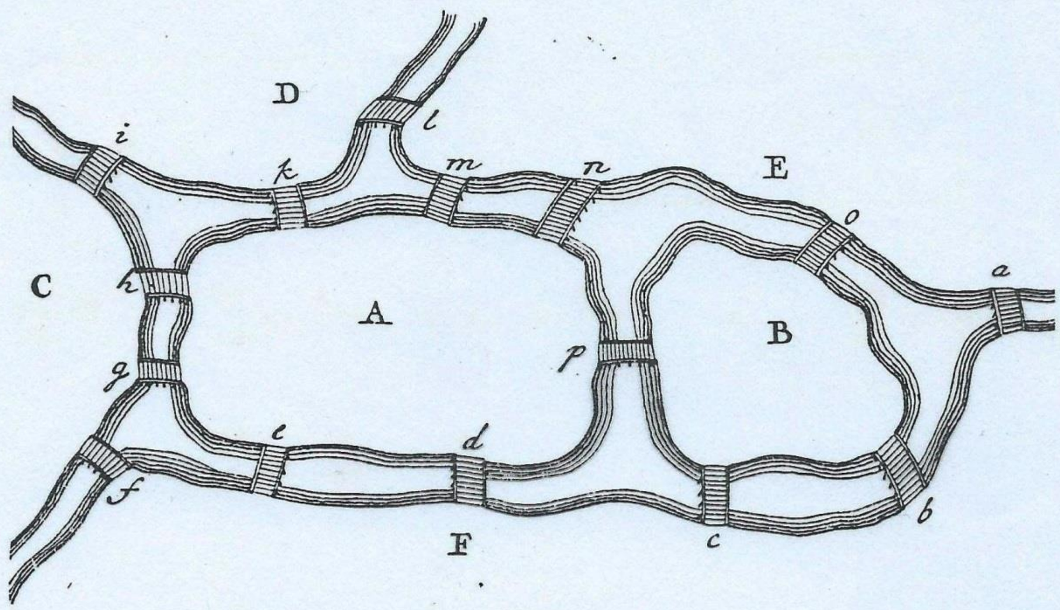


Fig. 3.

Euler's solution

20. Casu ergo quocunque proposito statim facillime poterit cognosci, utrum transitus per omnes pontes semel institui queat an non, ope huius regulae:

Si fuerint plures duabus regiones, ad quas ducentium pontium numerus est impar, tum certo affirmari potest talem transitum non dari.

Si autem ad duas tantum regiones ducentium pontium numerus est impar, tunc transitus fieri poterit, si modo cursus in altera harum regionum incipiatur.

Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum transitus desiderato modo institui poterit, in quacunque regione ambulandi initium ponatur.

Hac igitur data regula problemati proposito plenissime satisfit.

If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

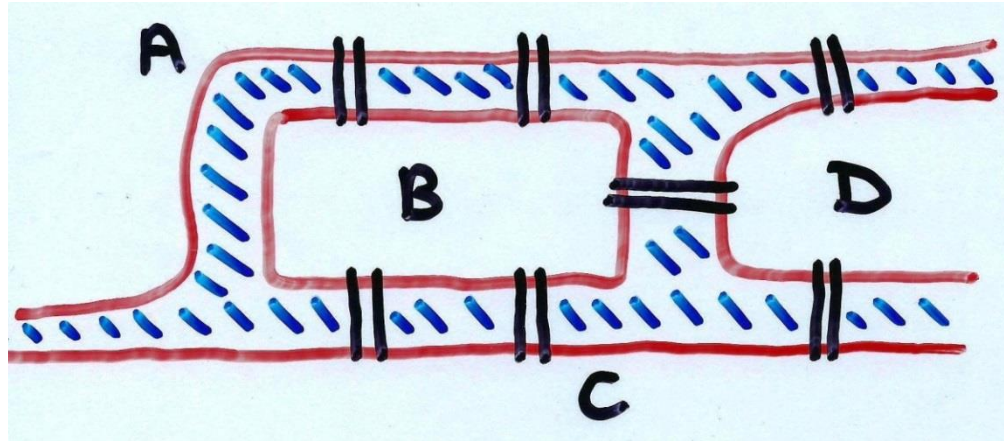
If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.

If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

So the Königsberg bridges problem has no solution.

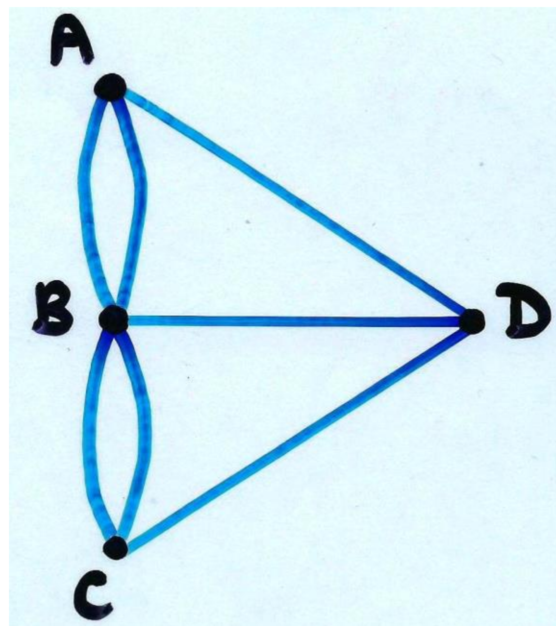
But Euler did not prove the sufficiency: this was first proved by C. Hierholzer, 1871

The modern approach (using graphs)



Can you draw this picture in one continuous stroke?

Yes, if and only if the number of vertices of odd degree is 0 or 2.



NOT DRAWN BY EULER:

First appearance in 1892

1741–1766 in Berlin

1744: Calculus of variations

1748: *Introductio in Analysin Infinitorum*

$e^{ix} = \cos x + i \sin x$ Functions

Conics & quadrics Partitions

1749: Theory of tides

Motion of the moon

1749/50: Vibrating strings

Differential equations Waves

1750: Polyhedron formula

1755: *Calculi Differentialis*

1759: Knight's tour problem

1760: Differential geometry

Euler's *Introductio in Analysin Infinitorum* (1748)

INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII. (1748)

Partitions of numbers

Leibniz introduced these 'divulsions of integers'
in a letter to Johann Bernoulli

Split a number into smaller ones

1 = 1 (1 way)   2 = 2 or 1 + 1 (2 ways)

3 = 3 or 2 + 1 or 1 + 1 + 1 (3 ways)

4 = 4 or 3+1 or 2+2 or 2+1+1 or 1+1+1+1 (5 ways)

5 = 5 or 4+1 or 3+2 or 3+1+1 or 2+2+1 or … or … (7 ways)

. . .

p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11,

p(10) = 42, p(20) = 627, p(30) = 5604, p(40) = 37338, . . . ,

p(200) = 3,972,999,029,388

Euler's Pentagonal Number Theorem

Look at the generating function (or 'washing line'):

$$F(x) = 1 + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \ldots$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \ldots$$

In the *Introductio* Euler proved that

$$F(x) = (1 - x)^{-1} \times (1 - x^2)^{-1} \times (1 - x^3)^{-1} \times (1 - x^4)^{-1} \times \ldots$$

$$= 1 / \{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \ldots\}$$

and later that

$$(1 - x) \times (1 - x^2) \times (1 - x^3) \times (1 - x^4) \times \ldots$$

$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots$$

(The exponents $k(3k \pm 1)/2$ are the 'pentagonal numbers')

Euler's Partition Formula

Multiplying these expressions together we get:

$$\{1 + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \ldots\}$$
$$\times \{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots\} = 1.$$

Isolating the term in x^n and rearranging the result, we get:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7)$$
$$+ p(n-12) + p(n-15) - \ldots .$$

So each successive partition number p(n)
can be calculated from the previous ones.

So p(11) = p(10) + p(9) – p(6) – p(4) = 42 + 30 – 11 – 5 = 56.

Euler calculated p(n) up to p(65) = 2012558.

This is still the most efficient way to calculate partition numbers.

Philip Naude's problems

In how many ways can 50 be written as the sum of seven distinct integers?

Euler: Consider $(1 + xz) \times (1 + x^2z) \times (1 + x^3z) \times (1 + x^4z) \times \; . \, . \, .$

$$= 1 + z\,(x + x^2 + x^3 + x^4 + \ldots) + z^2\,(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + \ldots) + z^3\,(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + \ldots) + \ldots$$

Answer = coefficient of x^{50} in the row $z^7\,(\ldots) = 522$.

What is the corresponding answer if the seven integers are not distinct?

Euler: Consider $(1 - xz)^{-1} \times (1 - x^2z)^{-1} \times (1 - x^3z)^{-1} \times (1 - x^4z)^{-1} \times \ldots$

$$= (1 + xz + x^2z^2 + x^3z^3 + \ldots) \times (1 + x^2z + x^4z^2 + \ldots) \times \ldots$$

Answer (after some calculation) = 8496.

Odd and distinct partitions

In odd partitions all the parts are odd

There are eight odd partitions of 9:

9, 7+1+1, 5+3+1, 5+1+1+1+1, 3+3+3, 3+3+1+1+1,
3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1

In distinct partitions all the parts are distinct

There are eight distinct partitions of 9:

9, 8+1, 7+2, 6+3, 6+2+1, 5+4, 5+3+1, 4+3+2

Euler found the following generating functions:

odd partitions: $(1-x)^{-1} \times (1-x^3)^{-1} \times (1-x^5)^{-1} \times (1-x^7)^{-1} \times \ldots$

distinct partitions: $(1+x) \times (1+x^2) \times (1+x^3) \times (1+x^4) \times \ldots$

and showed that they are equal:

For any positive integer, the number of odd partitions always equals the number of distinct partitions.

Partition numbers up to p(200), calculated by Percy MacMahon

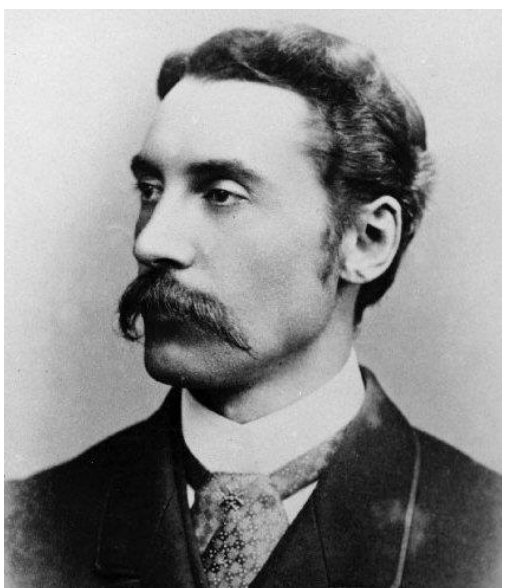


TABLE IV*: $p(n)$.

1...	1	51...	239943	101...	214481126	151...	45060624582
2...	2	52...	281589	102...	241265379	152...	49686288421
3...	3	53...	329931	103...	271248950	153...	54770336324
4...	5	54...	386155	104...	304801365	154...	60356673280
5...	7	55...	451276	105...	342325709	155...	66493182097
6...	11	56...	526823	106...	384276336	156...	73232243759
7...	15	57...	614154	107...	431149389	157...	80630964769
8...	22	58...	715220	108...	483502844	158...	88751778802
9...	30	59...	831820	109...	541946240	159...	97662728555
10...	42	60...	966467	110...	607163746	160...	107438159466
11...	56	61...	1121505	111...	679903203	161...	118159068427
12...	77	62...	1300156	112...	761002156	162...	129913904637
13...	101	63...	1505499	113...	851376628	163...	142798995930
14...	135	64...	1741630	114...	952050665	164...	156919475295
15...	176	65...	2012558	115...	1064144451	165...	172389800255
16...	231	66...	2323520	116...	1188908248	166...	189334822579
17...	297	67...	2679689	117...	1327710076	167...	207890420102
18...	385	68...	3087735	118...	1482074143	168...	228204732751
19...	490	69...	3554345	119...	1653668665	169...	250438925115
20...	627	70...	4087968	120...	1844349560	170...	274768617130
21...	792	71...	4697205	121...	2056148051	171...	301384802048
22...	1002	72...	5392783	122...	2291320912	172...	330495499613
23...	1255	73...	6185689	123...	2552338241	173...	362326859895
24...	1575	74...	7089500	124...	2841940500	174...	397125074750
25...	1958	75...	8118264	125...	3163127352	175...	435157697830
26...	2436	76...	9289091	126...	3519222692	176...	476715857290
27...	3010	77...	10619863	127...	3913864295	177...	522115831195
28...	3718	78...	12132164	128...	4351078600	178...	571701605655
29...	4565	79...	13848650	129...	4835271870	179...	625846753120
30...	5604	80...	15796476	130...	5371315400	180...	684957390936
31...	6842	81...	18004327	131...	5964539504	181...	749474411781
32...	8349	82...	20506255	132...	6620830889	182...	819876908323
33...	10143	83...	23338469	133...	7346629512	183...	896684817527
34...	12310	84...	26543660	134...	8149040695	184...	980462880430
35...	14883	85...	30167357	135...	9035836076	185...	1071823774337
36...	17977	86...	34262962	136...	10015581680	186...	1171432692373
37...	21637	87...	38887673	137...	11097645016	187...	1280011042268
38...	26015	88...	44108109	138...	12292341831	188...	1398341745571
39...	31185	89...	49995925	139...	13610949895	189...	1527273599625
40...	37338	90...	56634173	140...	15065878135	190...	1667727404093
41...	44583	91...	64112359	141...	16670689208	191...	1820701100652
42...	53174	92...	72533807	142...	18440293320	192...	1987276856363
43...	63261	93...	82010177	143...	20390982757	193...	2168627105469
44...	75175	94...	92669720	144...	22540654445	194...	2366022741845
45...	89134	95...	104651419	145...	24908858009	195...	2580840212973
46...	105558	96...	118114304	146...	27517052599	196...	2814570987591
47...	124754	97...	133230930	147...	30388671978	197...	3068829878530
48...	147273	98...	150198136	148...	33549419497	198...	3345365983698
49...	173525	99...	169229875	149...	37027355200	199...	3646072432125
50...	204226	100...	190569292	150...	40853235313	200...	3972999029388

Hardy & Ramanujan on partitions

Statement of the main theorem.

THEOREM. *Suppose that*

(1·71) $$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right),$$

where C and λ_n are defined by the equations (1·53), *for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula*

(1·721)
$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\tfrac14(2-pq-p)+\tfrac1{12}\left(q-\frac1q\right)(2p-p'+p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

(1·722)
$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\tfrac14(q-1)+\tfrac1{12}\left(q-\frac1q\right)(2p-p'+p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi†, and p' is any positive integer such that $1+pp'$ is divisible by q; that

(1·73) $$A_q(n) = \sum_{(p)} \omega_{p,q} e^{-2np\pi i/q};$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

Then

(1·74) $$p(n) = \sum_1^{\nu} A_q\phi_q + O(n^{-\frac14}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

(1·75) $$\sum_1^{\nu} A_q\phi_q.$$

Euler's polyhedron formula:

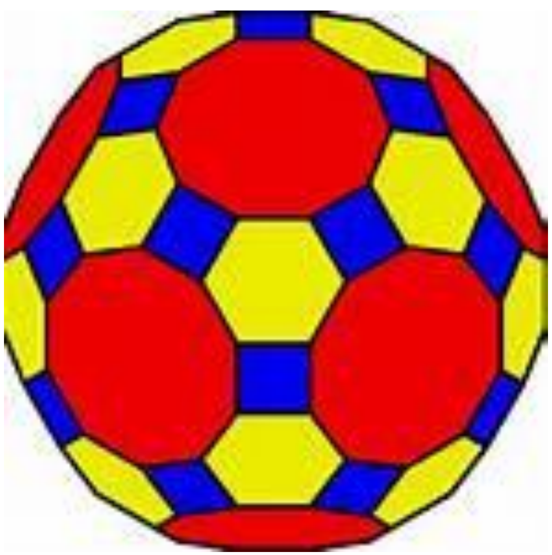
F + V = E + 2

cube

6 faces, 8 vertices,
12 edges
and 6 + 8 = 12 + 2

dodecahedron

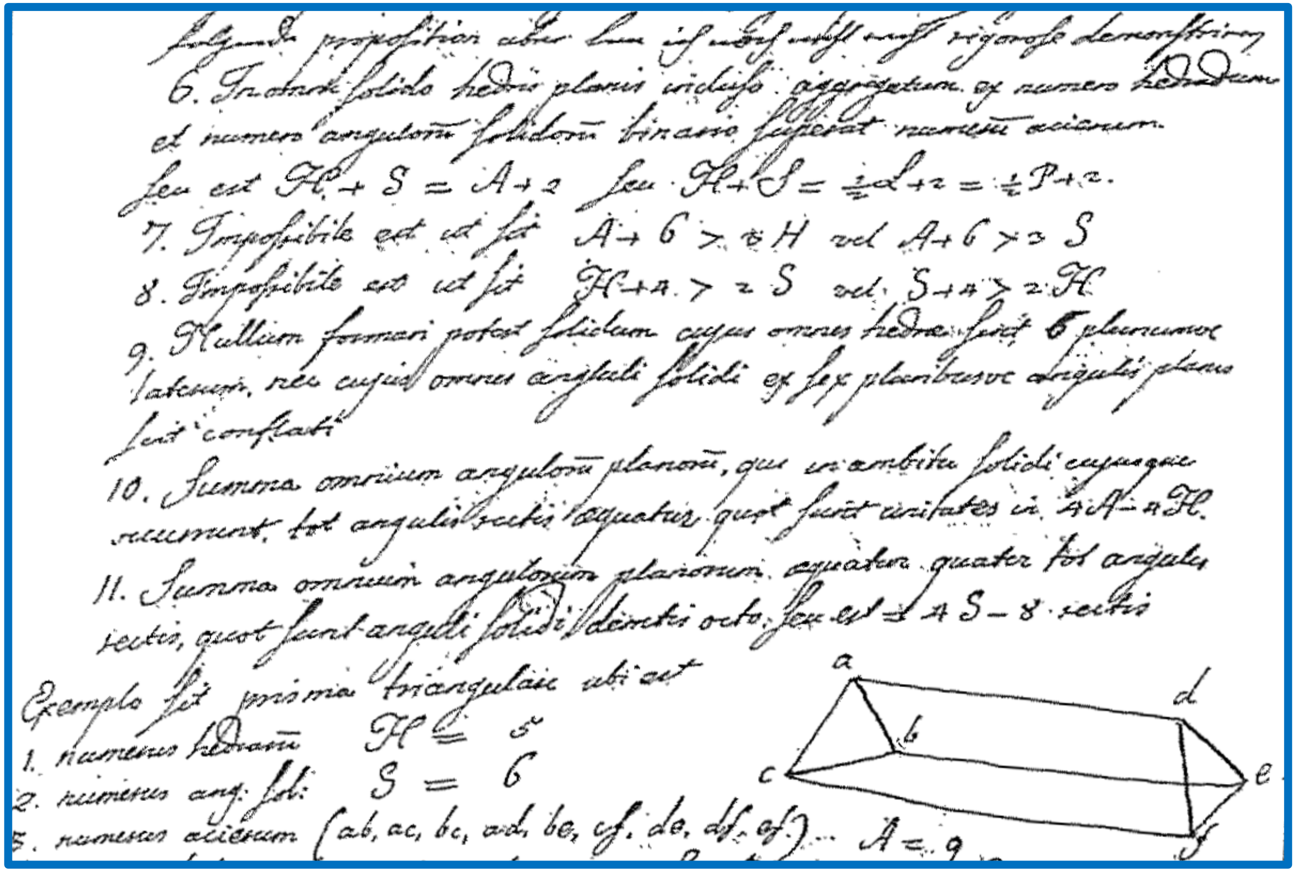
12 faces, 20 vertices,
30 edges
and 12 + 20 = 30 + 2



great rhombicosidodecahedron

62 faces, 120 vertices, 180 edges
and 62 + 120 = 180 + 2

Euler's letter to C. Goldbach (1750)



folgende propositiones aber kan ich noch nicht recht rigorose demonstriren

6. In omni solido hedris planis incluso aggregatum ex numero hedrarum et numero angulorum solidorum binario superat numerum acierum. seu est $H + S = A + 2$ seu $H + S = \frac{1}{2}L + 2 = \frac{1}{4}P + 2$.

7. Impossibile est ut sit $A + 6 > 3H$ vel $A + 6 > 3S$

8. Impossibile est ut sit $H + 4 > 2S$ vel $S + 4 > 2H$

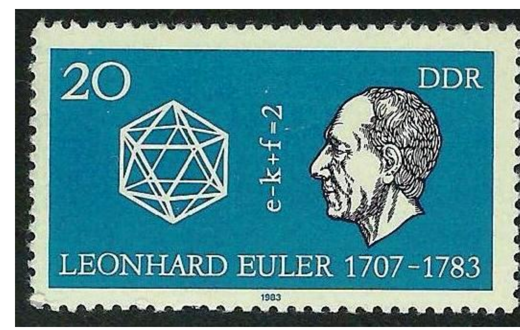
9. Nullum formari potest solidum cujus omnes hedrae sint 6 pluriumve laterum, nec cujus omnes anguli solidi ex sex pluribusve angulis planis sint conflati

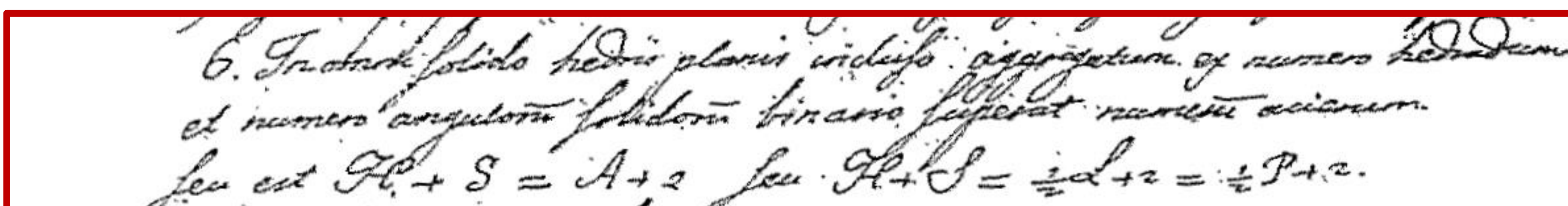
10. Summa omnium angulorum planorum, qui in ambitu solidi cujusque occurrunt, tot angulis rectis aequatur quot sunt unitates in $4A - 4H$.

11. Summa omnium angulorum planorum aequatur quater tot angulis rectis, quot sunt anguli solidi demtis octo, seu est $= 4S - 8$ rectis

Exempli sit prisma triangulare ubi est

1. numerus hedrarum $H = 5$
2. numerus ang. sol. $S = 6$
3. numerus acierum (ab, ac, bc, ad, be, cf, de, df, ef) $A = 9$





6. In omni solido hedris planis incluso aggregatum ex numero hedrarum et numero angulorum solidorum binario superat numerum acierum. seu est $H + S = A + 2$ seu $H + S = \frac{1}{2}L + 2 = \frac{1}{4}P + 2$.

Euler's 1750 letter

6. In every solid enclosed by plane faces, the aggregate of the number of faces and the number of solid angles exceeds by 2 the number of edges, or $H + S = A + 2$.

H = *hedrae* (faces); S = *angulae solidae* (solid angles = vertices),
A = *acies* (edges) – a term due to Euler

11. The sum of all the plane angles is equal to four times as many right angles as there are solid angles, less 8 – that is, $= 4S - 8$ right angles.

I find it surprising that these general results in solid geometry have not previously been noticed by anyone, as far as I am aware; and furthermore, that the important ones, Theorems 6 and 11, are so difficult that I have not yet been able to prove them in a satisfactory way.

Proving the polyhedron formula

In 1752 Euler tried to prove the polyhedron formula by slicing corners off the polyhedron in such a way that $S - A + H$ remains unchanged at each stage, until a tetrahedron was reached (with $S - A + H = 4 - 6 + 4 = 2$), but his proof was deficient.

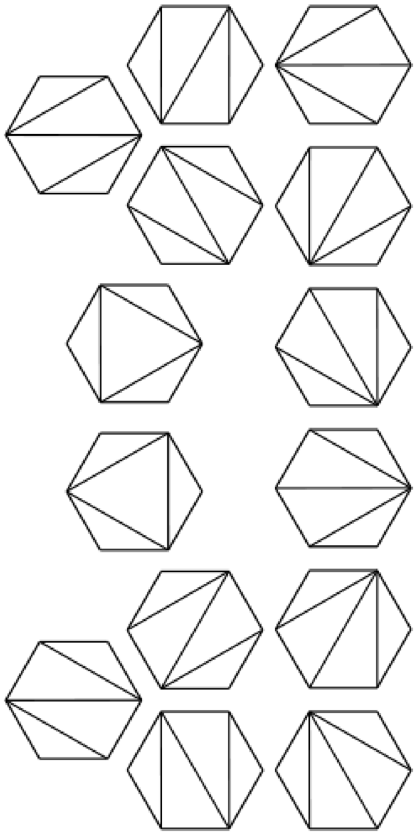
The first correct proof was a metrical one given by A.-M. Legendre in 1794

Later proofs were given in the 1810s by A.-L. Cauchy and S.-A.-J. L'huilier.

Dividing polygons (1751)

In how many ways can a regular n-sided polygon be divided into triangles?

For $n = 6$ there are 14 ways (shown), and for $n = 10$ there are 1430 ways.



Euler proved that the number of ways is

$$2 \times 6 \times 10 \times \ldots \times (4n - 10) \,/\, (n - 1)!$$

(so, for $n = 6$, we have $2 \times 6 \times 10 \times 14 / 120 = 14$)

and that the generating function is

$$x^3 + 2x^4 + 5x^5 + 14x^6 + 42x^7 + 132x^8 + \ldots$$
$$= x\,\{1 - 2x - \sqrt{(1 - 4x)}\} \,/\, 2.$$

These numbers were later called *Catalan numbers*, after Eugène Catalan, who wrote about them in 1838.

Derangement problem (1753)

Two players turn over identical packs of cards, one card at a time. The first player wins if there's a 'match'. What is the probability that no match occurs?

In how many ways (D_n) can n given letters be arranged so that none is in its original position?

For example, if n = 4, there are 9 (out of 24) possible ways:

badc, bcda, bdac, cadb, cdab, cdba, dabc, dcab, dcba.

n	1	2	3	4	5	6	7	8	...
$n!$	1	2	6	24	120	720	5040	40320	...
D_n	0	1	2	9	44	265	1854	14833	...
$D_n / n!$	0	0.5	0.3333	0.375	0.3667	0.3681	0.3678	0.3679	...

Solving the derangement problem

In how many ways (D_n) can n given letters be arranged so that none is in its original position?

Around 1710 the derangement problem had been solved by De Moivre and de Montmort. Euler revisited the problem:

Euler :

$$D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$$

$$D_n = nD_{n-1} + (-1)^n$$

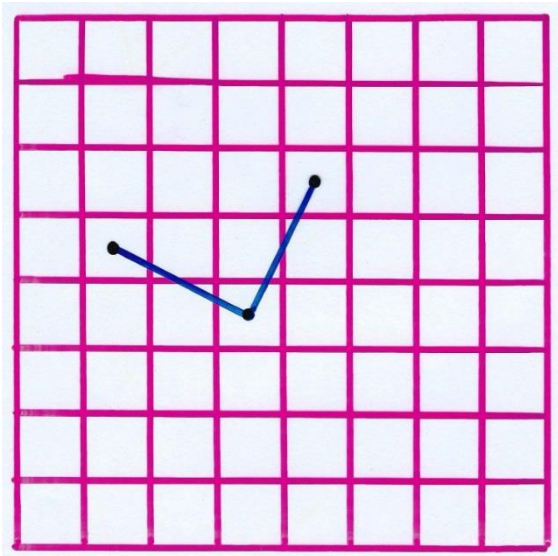
$$\Rightarrow \quad D_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!}\right\}$$

$$\approx n!/e$$

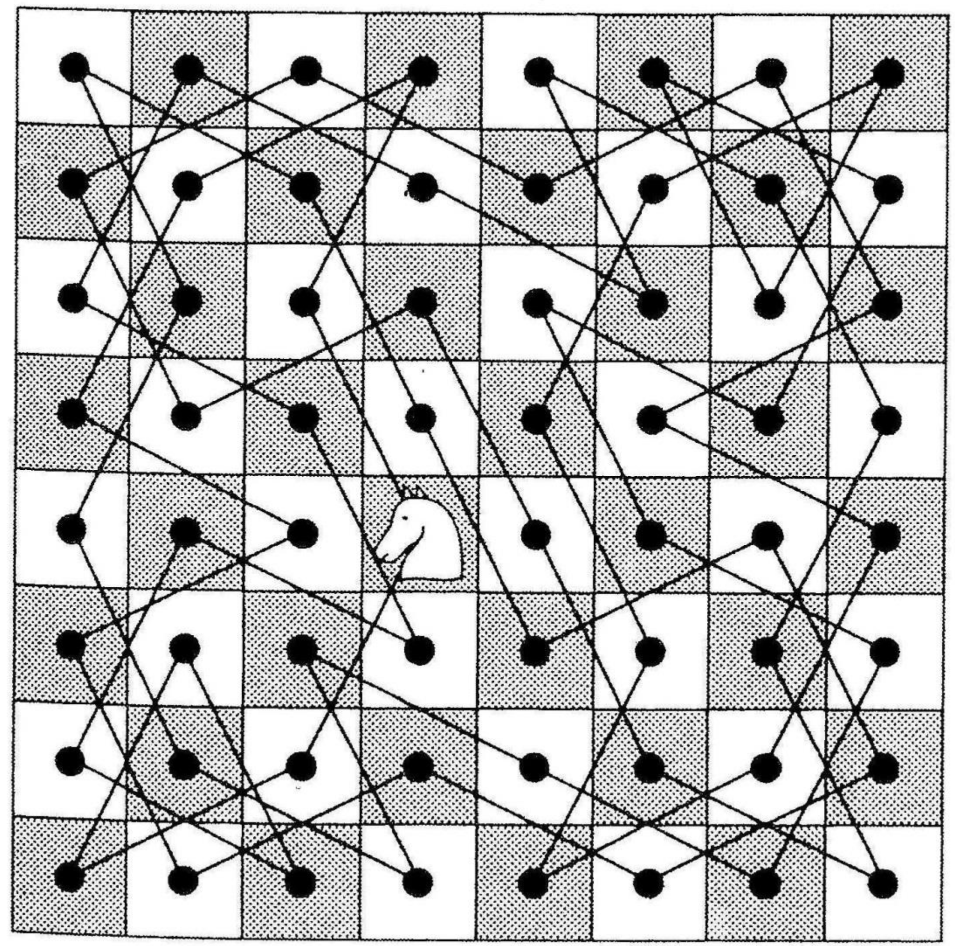
In fact, D_n is always the nearest integer to n!/e.

For example, if n = 8, D_n = 14833 and n!/e ≈ 14832.9.

So in the card problem, the probability of no match = 1/e ≈ 0.368.



Knight's-tour problem (1759)

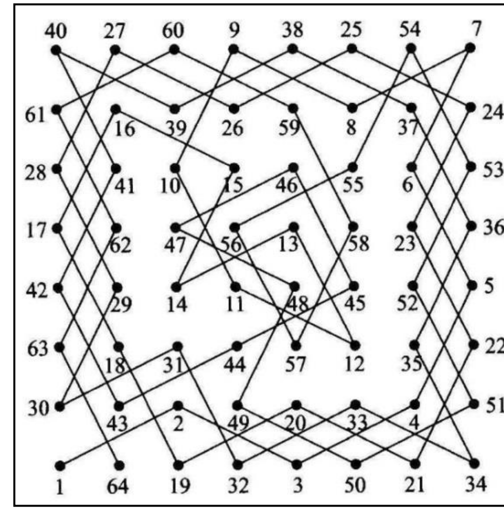


Can a knight visit all the squares of a chessboard by knight's moves and return to its starting point?

Knight’s-tour problem

Euler gave the first systematic treatment of the problem, exhibiting several solutions with various degrees of symmetry.

As he observed, there is no knight’s tour on an n × n chessboard when n is odd (since a knight must ‘alternate colours’), and he gave several examples when n = 6, 8 and 10.



30	41	46	37	32	53	60	67	72	55
47	36	31	40	45	68	73	54	61	66
42	29	38	33	50	59	52	63	56	71
35	48	27	44	39	74	69	58	65	62
28	43	34	49	26	51	64	75	70	57
7	20	25	14	1	76	99	84	93	78
12	15	8	19	24	89	94	77	98	85
21	6	13	2	9	100	83	88	79	92
16	11	4	23	18	95	90	81	86	97
5	22	17	10	3	82	87	96	91	80

1766–1783 in St Petersburg

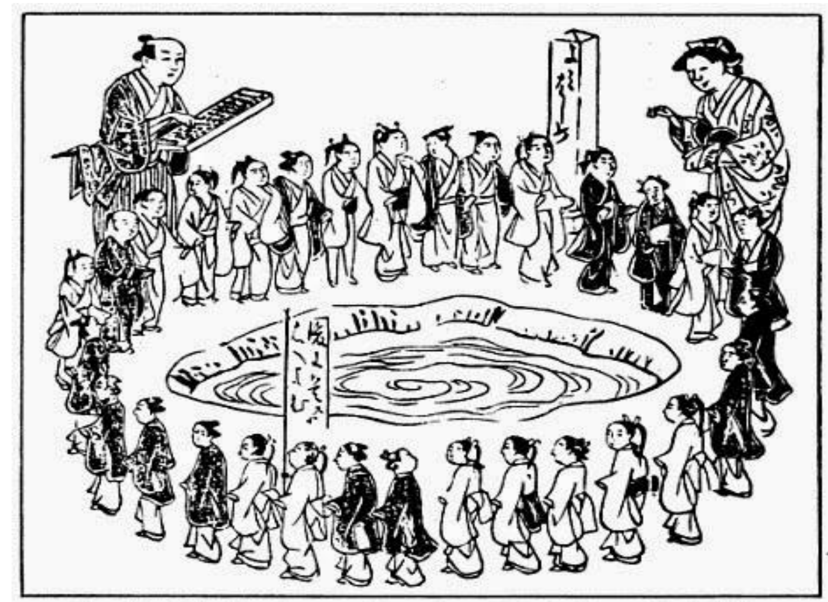
1767: Euler line of a triangle
1768/74: Letters to a German Princess
1768–70: *Calculi Integralis* (3 volumes)
1770: Algebra / number theory
1771: Dioptrica (optics)
1773: Sailing of ships
1774: Astronomy book
1776: Motion of rigid bodies
1776: 775-page treatise on the motion of the moon
1782: Magic and Latin squares / 36 Officers problem
1783: Died 7/18 September

The Josephus problem (1771)

Suppose that n people stand in a circle.
Moving clockwise, we eliminate every kth person.
How do you ensure that you are the last to go?

Named after Flavius Josephus, who was imprisoned by the Romans in the 1st century.

For example, with n = 15 and k = 4, we eliminate 4, 8, 12, 1, 6, 11, 2, 9, 15, 10, 5, 3, 7, 14, 13.



Japanese print from 1797

Euler developed a procedure for solving this problem, showing that, when n = 5000 and k = 9, the survivor is 4897.

The death of Euler (1783)

On the 7th of September 1783, after amusing himself with calculating on a slate the laws of the ascending motion of air balloons, the recent discovery of which was then making a noise all over Europe, he dined with Mr Lexell and his family, talked of Herschel's planet (Uranus), and of the calculations which determine its orbit.

A little after, he called his grandchild, and fell a playing with him as he drank tea, when suddenly the pipe, which he held in his hand, dropped from it, and he ceased to calculate and to breathe.