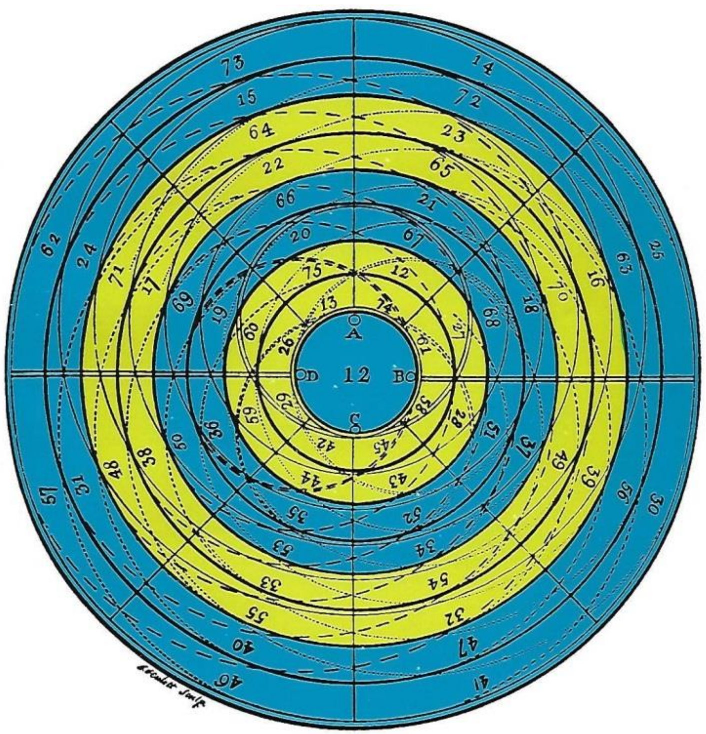
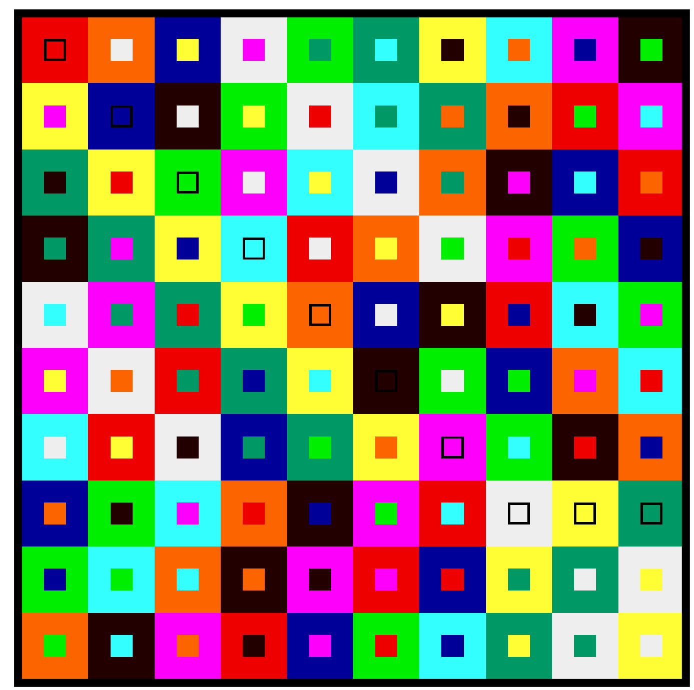
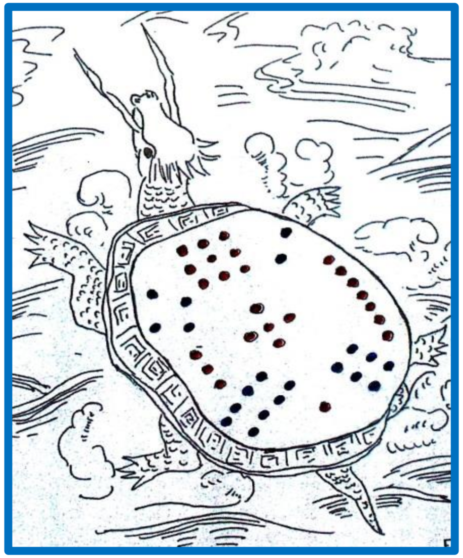
4. Magic Squares, Latin Squares and Triple Systems

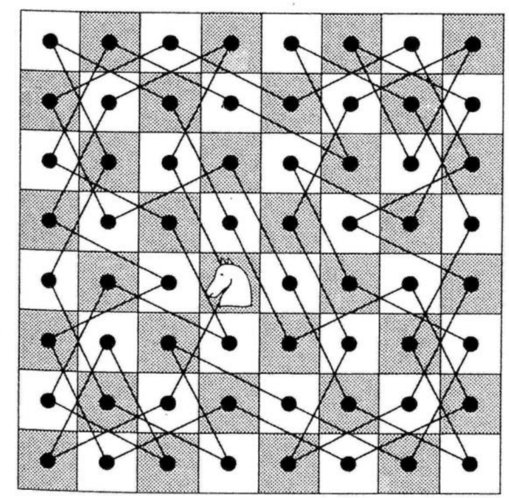
Robin Wilson

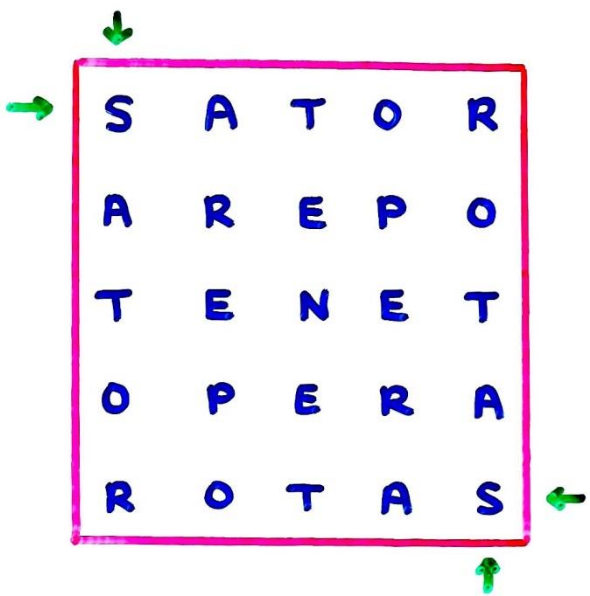


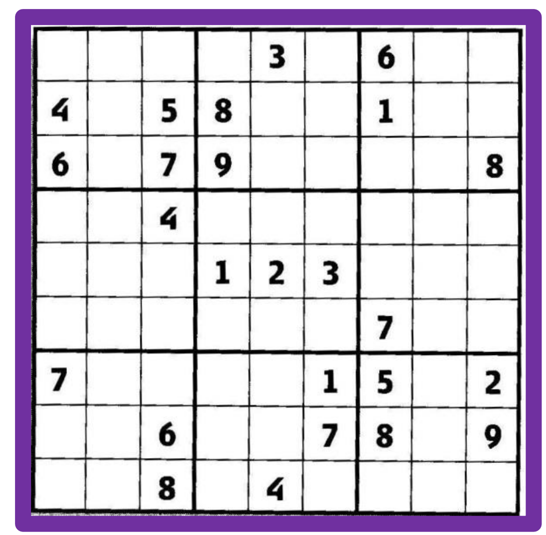


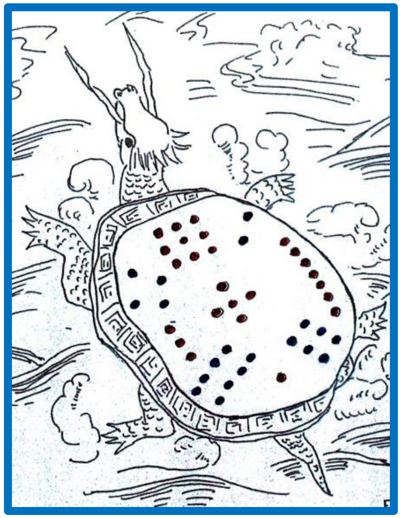
Square patterns

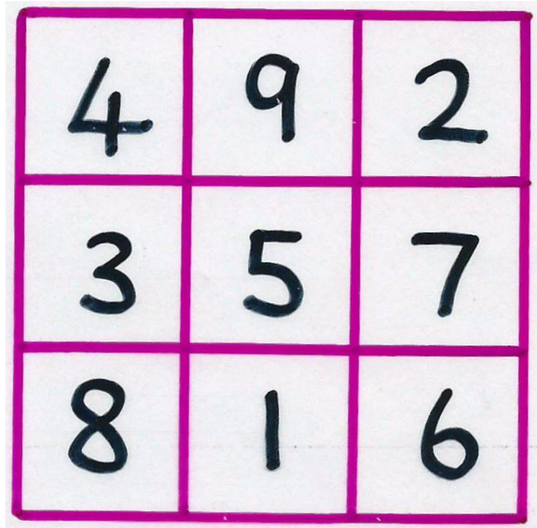




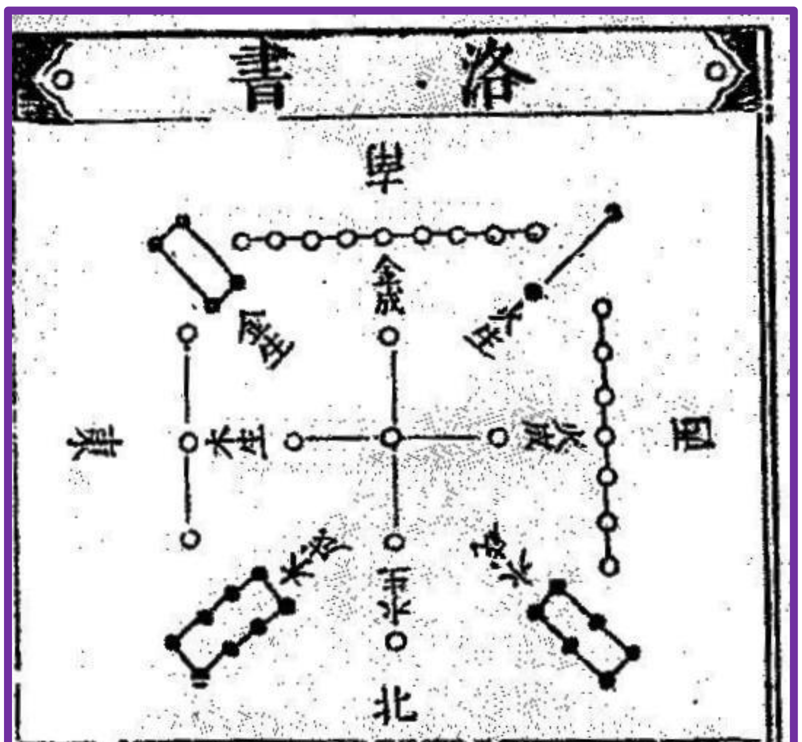








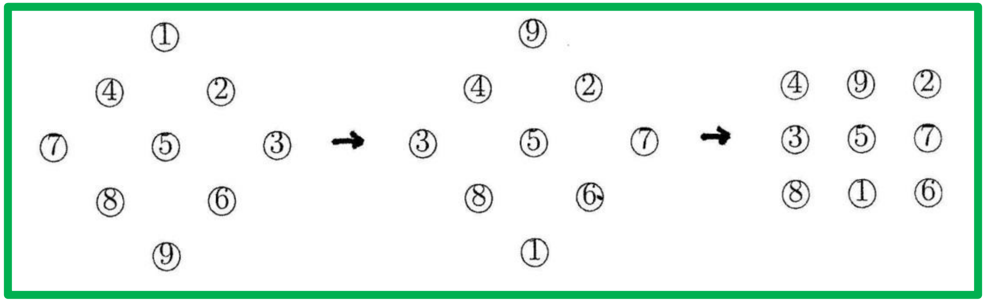
The Lo-shu diagram



The Lo-shu had magical significance – for example, relating to nine halls of a mythical palace where rites were performed in the 1st century AD

Yang Hui (c. 1238-98)

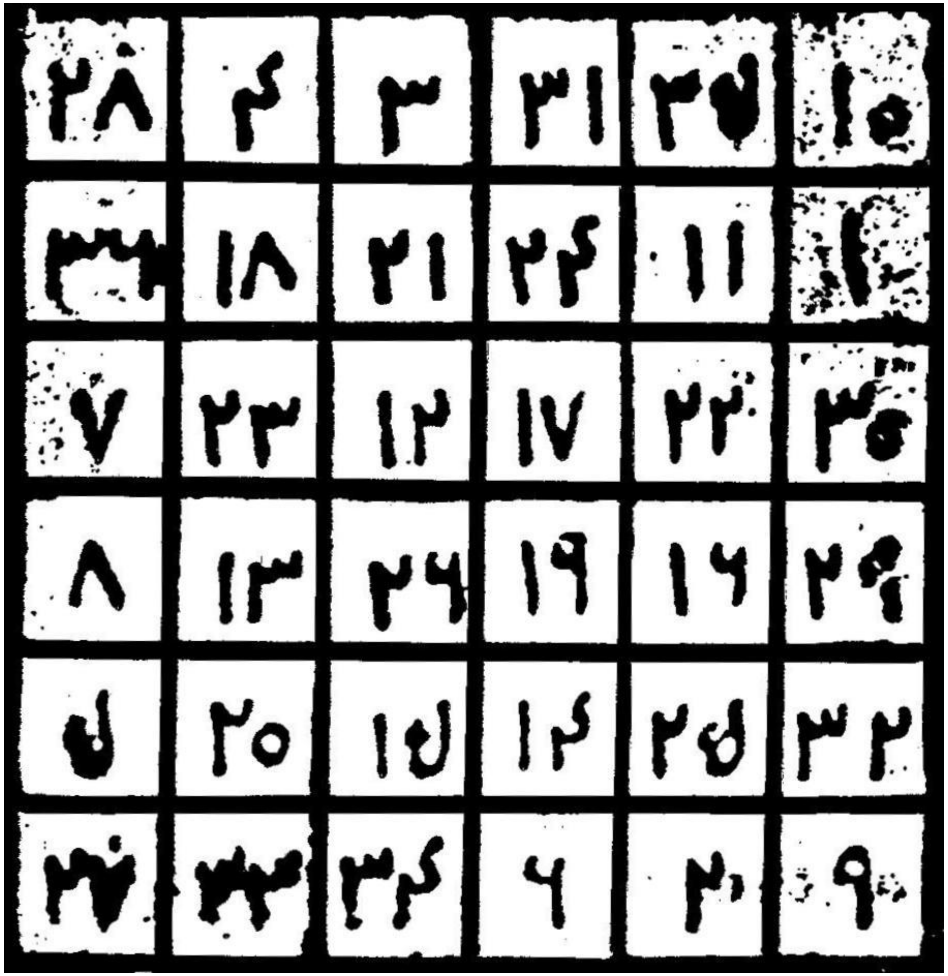
Yang Hui constructed a range of magic squares of different sizes, and with different properties.



46	8	16	20	29	7	49
3	40	35	36	18	41	2
44	12	33	23	19	38	6
28	26	11	25	39	24	22
5	37	31	27	17	13	45
48	9	15	14	32	10	47
1	43	34	30	21	42	4

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	4	49	72	9	54	65	2	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	3	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

Iron plate found at Xian (c. 1300)



28	4	3	31	35	10
36	18	21	24	11	1
7	23	12	17	22	30
8	13	26	19	16	29
5	20	15	14	25	32
27	33	34	6	2	9

Arabic (and later) magic squares

990: Ikhwan-al-Safa (Brethren of Purity) gave simple constructions for magic squares of sizes 3 – 6 (and possibly from 7 – 9), but with no general rule.

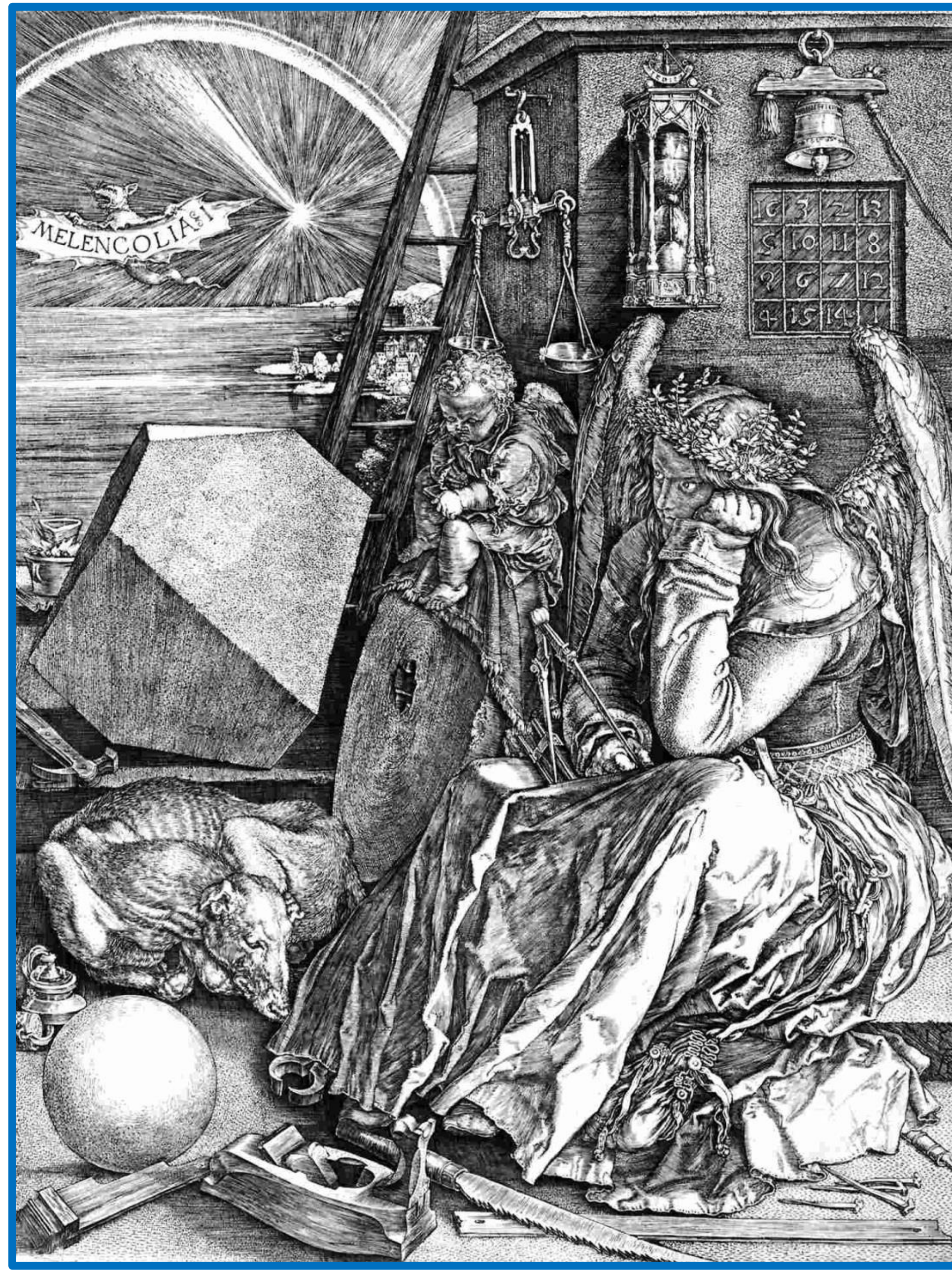
1200 al-Buni described a ‘bordering technique’.

1315 Moschopoulos gave general rules for constructing n × n magic squares when n is odd, or when n is divisible by 4.

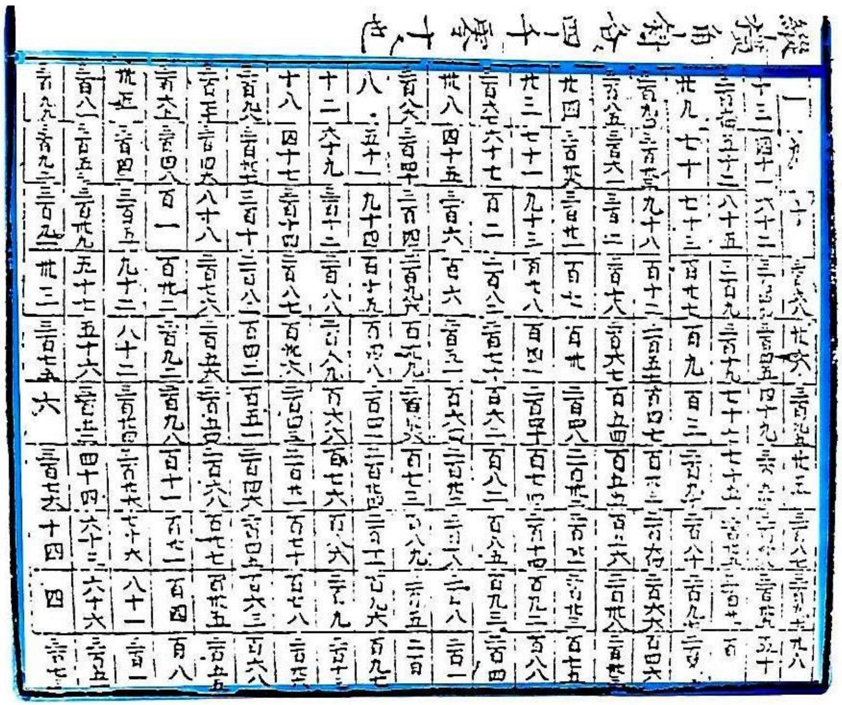
1691: Simon de la Loubère brought to France a simple method of Siamese origin for constructing magic squares when n is odd.

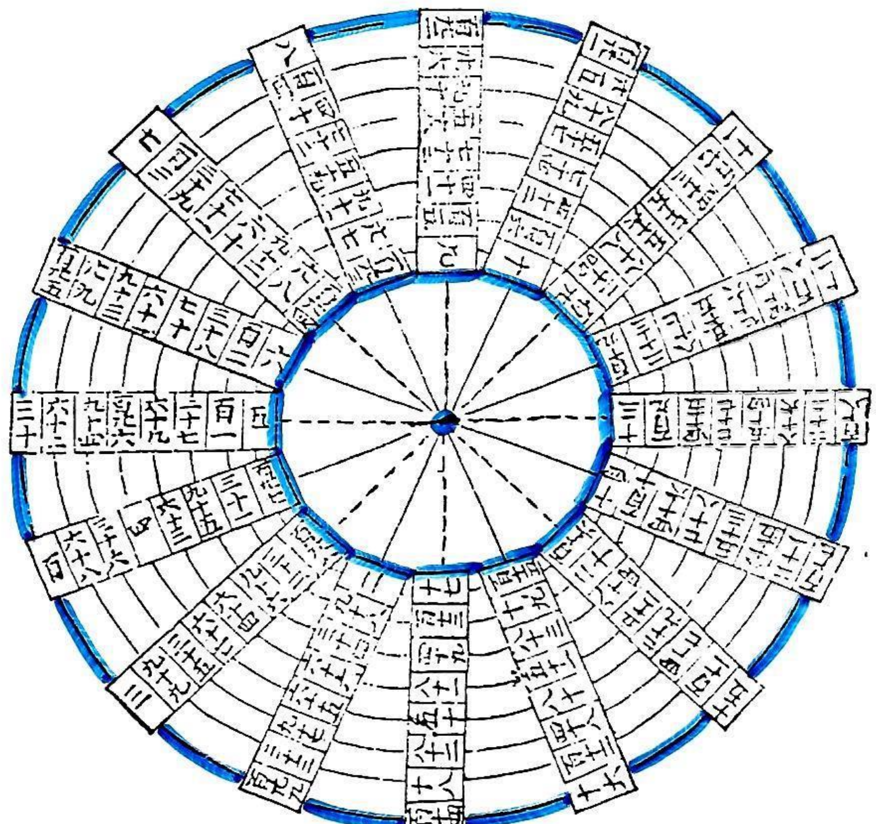
1693 Frenicle de Bessy obtained all 880 4 × 4 magic squares.

Dürer's *Melencholia 1* (1514)

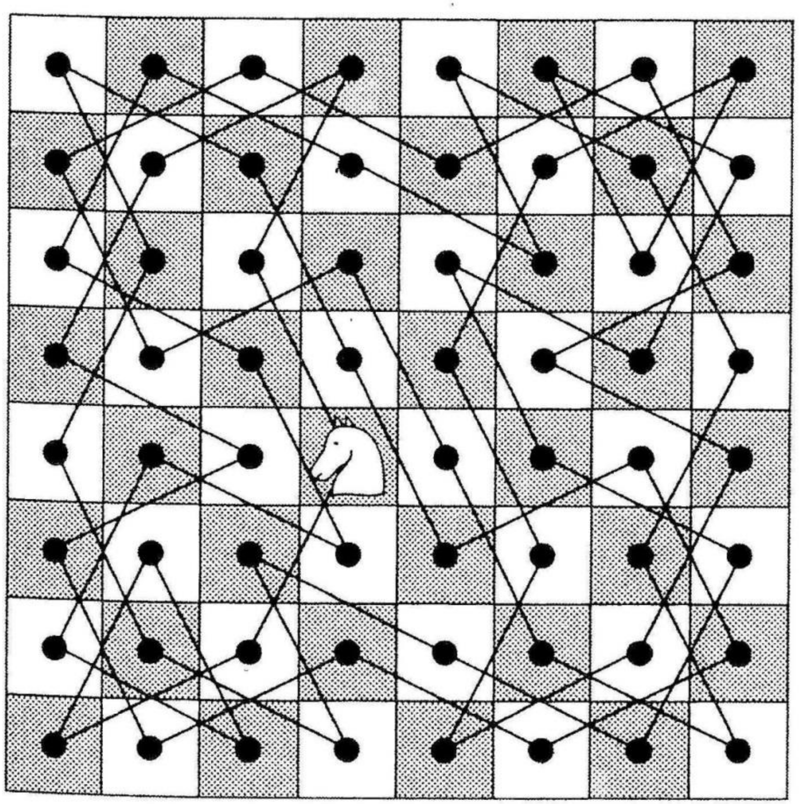


17th century Japanese magic figures





A knight's-tour 'magic square'



50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	64	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	1	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

Magic square of al-Antaakii (d. 987)

62	2	222	220	8	10	214	213	212	16	18	206	204	24	64
126	78	26	198	196	32	11	189	207	34	190	188	40	80	100
128	122	94	42	182	7	35	173	183	203	180	48	96	104	98
50	124	118	110	3	31	51	165	167	179	199	112	108	102	176
52	70	120	201	75	159	155	153	83	87	79	25	106	156	174
54	72	205	181	141	95	135	133	103	99	85	45	21	154	172
170	209	185	169	145	125	111	121	107	101	81	57	41	17	56
211	187	171	163	149	129	109	113	117	97	77	63	55	39	15
168	9	33	49	69	89	119	105	115	137	157	177	193	217	58
60	82	5	29	65	127	91	93	123	131	161	197	221	144	166
66	142	90	1	147	67	71	73	143	139	151	225	136	84	160
158	140	92	114	223	195	175	61	59	47	27	116	134	86	68
152	88	130	184	44	219	191	53	43	23	46	178	132	138	74
76	146	200	28	30	194	215	37	19	192	36	38	186	148	154
162	224	4	6	218	216	12	13	14	210	208	20	22	202	164

‘I was at length tired with sitting there to hear debates in which, as clerk, I could take no part, and which were often so unentertaining that I was induc’d to amuse myself with making magic squares …’.

Benjamin Franklin’s amazing 16 × 16 square

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

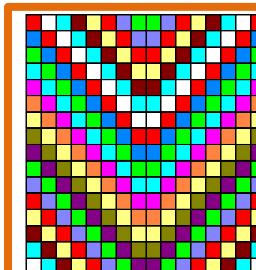


Figure 1

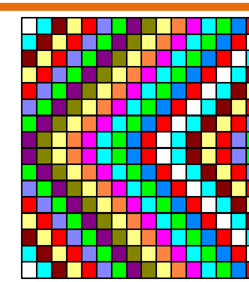


Figure 2

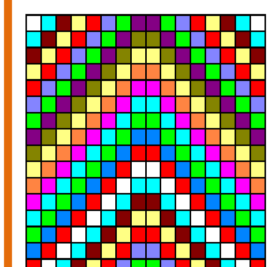


Figure 3

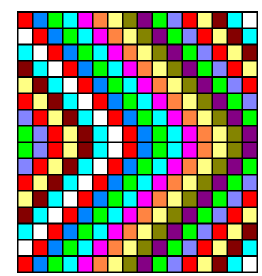


Figure 4

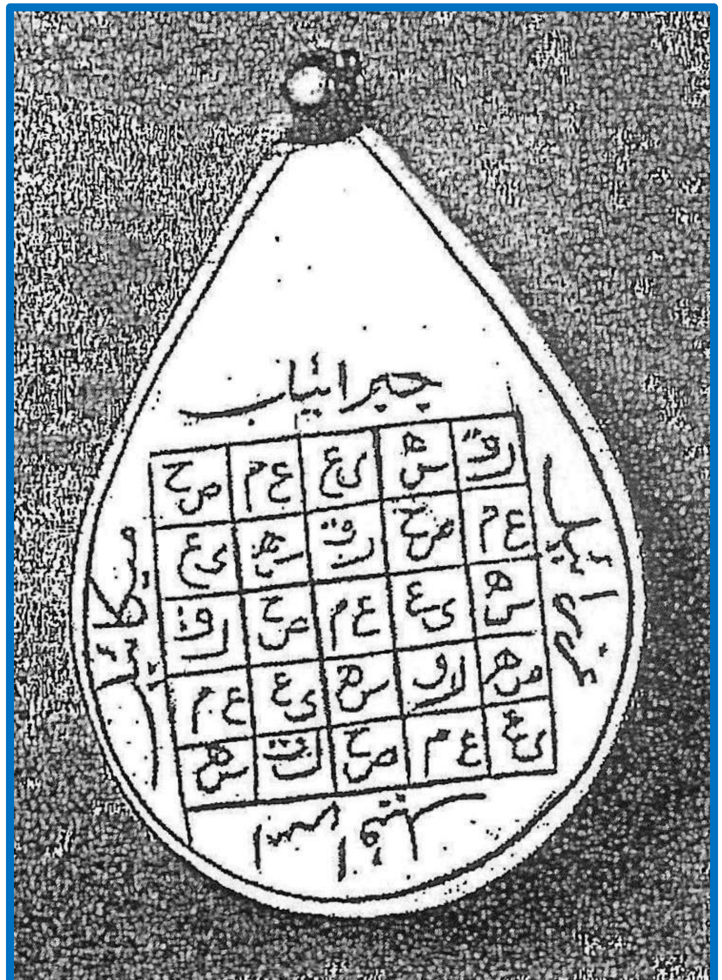
Three latin squares (3 × 3, 4 × 4, 5 × 5)

1	2	3
2	3	1
3	1	2

IGNIS			
IGNIS	AER	AQVA	TERRA
AER	IGNIS	TERRA	AQVA
AQVA	TERRA	IGNIS	AER
TERRA	AQVA	AER	IGNIS

فلان	الرحيم	الرحمن	الله	بسم
بسم	فلان	الرحيم	الرحمن	الله
الله	بسم	فلان	الرحيم	الرحمن
الرحمن	الله	بسم	فلان	الرحيم
الرحيم	الرحمن	الله	بسم	فلان

Silver amulet (Damascus, AD 1000?)



Two 7 × 7 latin squares

حرف الظاء للمشتري وله يوم الخميس

ظ	ث	ج	ف	خ	ش	ظ
ج	ف	خ	ش	ظ	ز	ث
خ	ش	ظ	ز	ث	ج	ف
ظ	ز	ث	ج	ف	خ	ش
ث	ج	ف	خ	ش	ظ	ز
ف	خ	ش	ظ	ز	ث	ج
ش	ظ	ز	ث	ج	ف	خ

al-Buni (c. 1200)

An example from 1788

MÉMOIRE

Sur les Avantages & l'Économie que procurent les racines employées à l'engrais des moutons à l'étable.

Par M. Cretté de Palluel.

EXPERIMENT UPON FATTING SHEEP, AND THEIR INCREASE FROM MONTH TO MONTH.

Sixteen ſheep, of the ſame age, of four different breeds, were picked out of my flock, viz. four the breed of the country, four of Beauce, four of Champagne, and four of Picardy; I weighed them alive, and marked each with a number; I divided them into four lots, and fed them on four different ſorts of food, as under:

Food.	No.	Breeds.	Weights at different Periods.—1788.					Increaſe each Month.				Total incr. which each food has produced upon four Sheep.
			Jan. 20.	Feb. 20.	Mar. 20.	April 20.	May 20.	1ſt M.	2d M.	3d M.	4th M.	
Potatoes,	1	Iſle de France,	69¾ lb.	79¼ lb.	—	—	—	10 lb.	lb.	lb.	lb.	70 lb.
	2	Beauce,	70¼	82½	90¼ lb.	93 lb.	95 lb.	11¼	7¼	2¼	2	
	3	Champagne,	69¼	83	82½	84	—	13¾	loſs ½	1½	—	
	4	Picardy,	88	95	101	—	—	15	6	—	—	
								50½	13½	4¼	2	
Turnips,	5	Iſle de France,	69	86	87	—	—	17	1	—	—	67½
	6	Beauce,	71	86	—	—	—	15	—	—	—	
	7	Champagne,	68½	78½	82½	84	84½	10	4	1½	½	
	8	Picardy,	79	95½	97½	97½	—	16½	2	—	—	
								58½	7	1½	½	
Beets,	9	Iſle de France,	72	83¼	90¼	94	—	11¼	7¼	3½	—	71
	10	Beauce,	70¼	80¼	86	—	—	10	5¼	—	—	
	11	Champagne,	77¼	90½	—	—	—	13¼	—	—	—	
	12	Picardy,	80	93½	98½	100½	101	13½	5	1½	½	
								48	17½	5	½	
Oats, Barley, and grey peas.	13	Iſle de France,	74	91	95½	102	106	17	4½	6½	4	92½
	14	Beauce,	73½	84¼	91½	96	—	10¼	7¼	4½	—	
	15	Champagne,	71	86¼	93	—	—	15¼	6¼	—	—	
	16	Picardy,	71	87	—	—	—	16	—	—	—	
								59	18½	11	4	

Observation. The increaſe of theſe ſheep, during the firſt month, being ſo much more conſiderable than in the following months, muſt be attributed to this cauſe, that lean cattle put up to fatten, eat greedily until they are cloyed, which only fills them, without much increaſing their fleſh; but, on the contrary, the increaſe produced in the enſuing months, although apparently leſs, turns all to profit in fleſh and tallow.

Sudoku puzzles

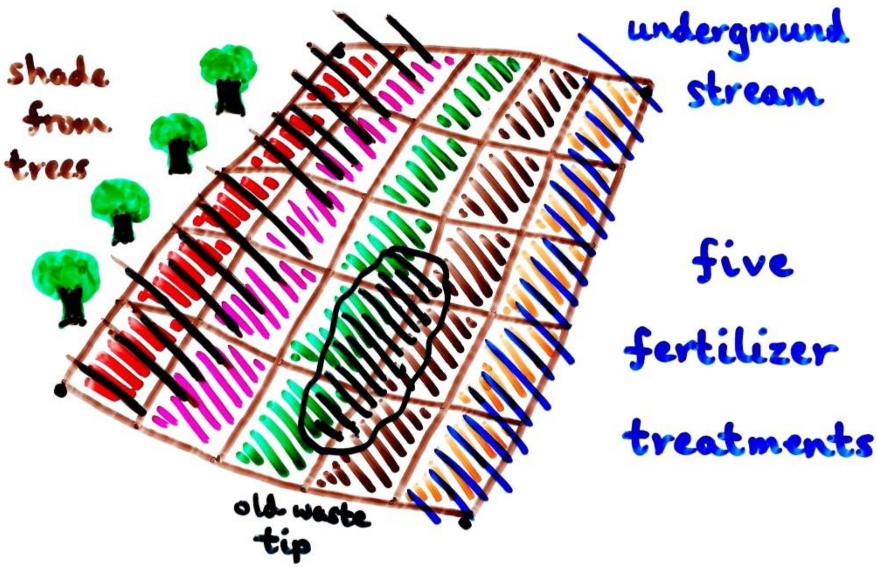
(1978 . . .)

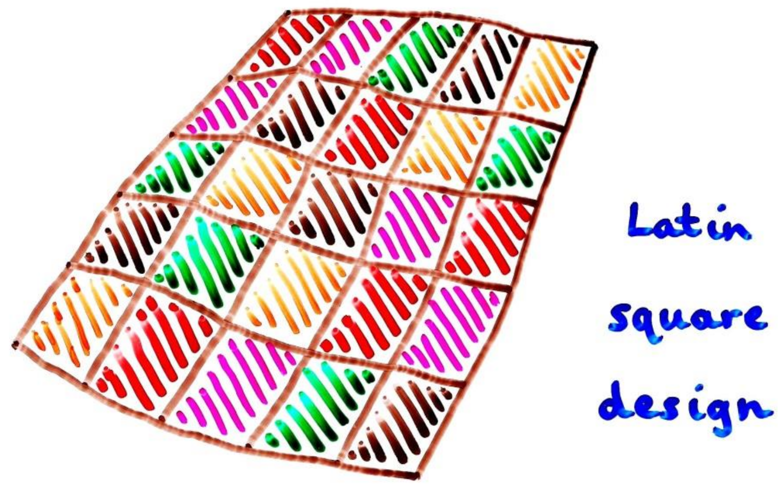
				3		6		
4		5	8			1		
6		7	9					8
		4						
			1	2	3			
						7		
7					1	5		2
		6			7	8		9
		8		4				

9	8	1	5	3	2	6	7	4
4	2	5	8	7	6	1	9	3
6	3	7	9	1	4	2	5	8
3	1	4	7	6	8	9	2	5
5	7	9	1	2	3	4	8	6
8	6	2	4	9	5	7	3	1
7	9	3	6	8	1	5	4	2
2	4	6	3	5	7	8	1	9
1	5	8	2	4	9	3	6	7

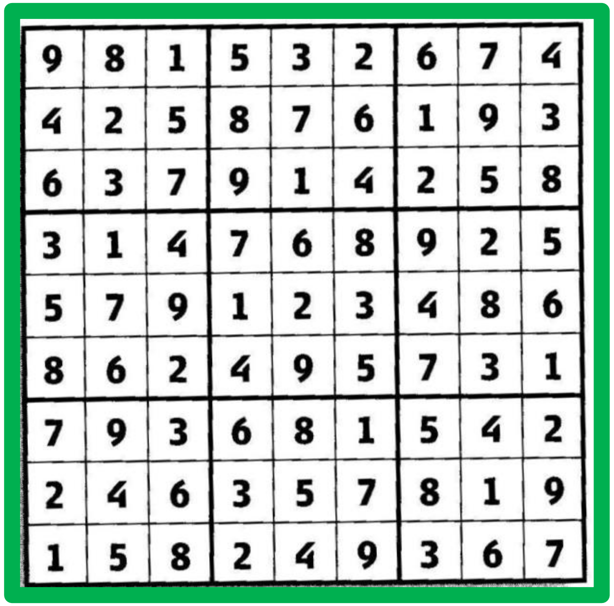
Design of experiments (1930s)
R. A. Fisher and F. Yates

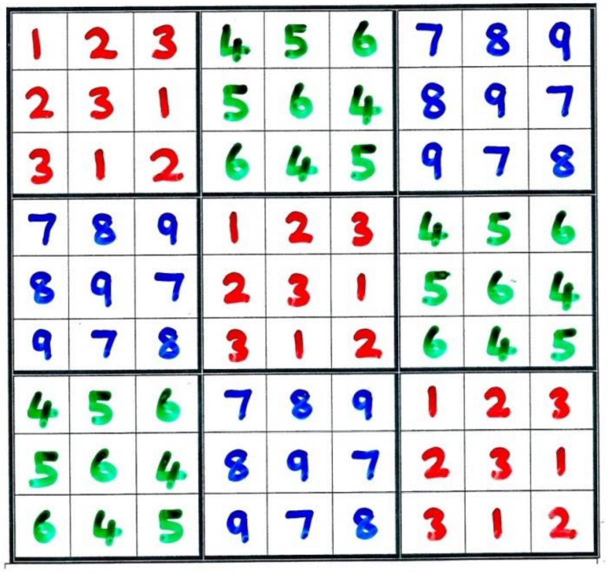
Latin squares in agriculture (design of experiments)

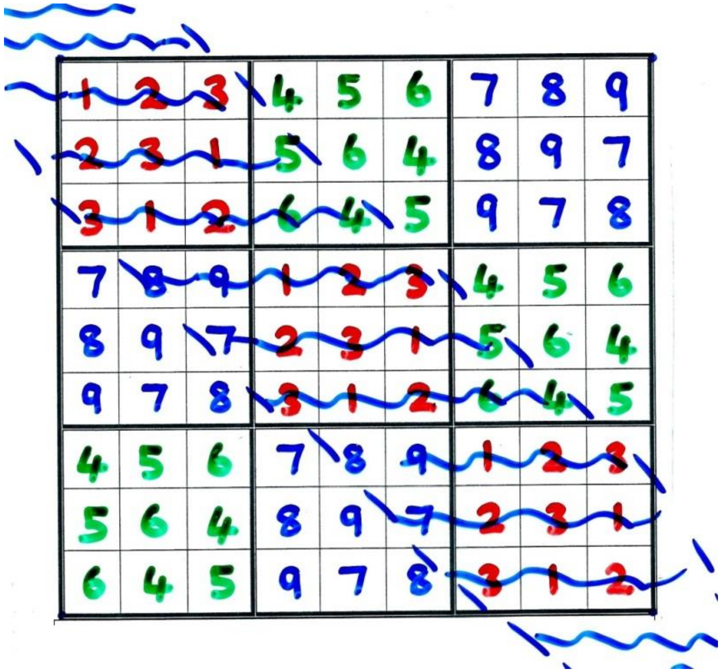




Sudoku designs







Court-card puzzle

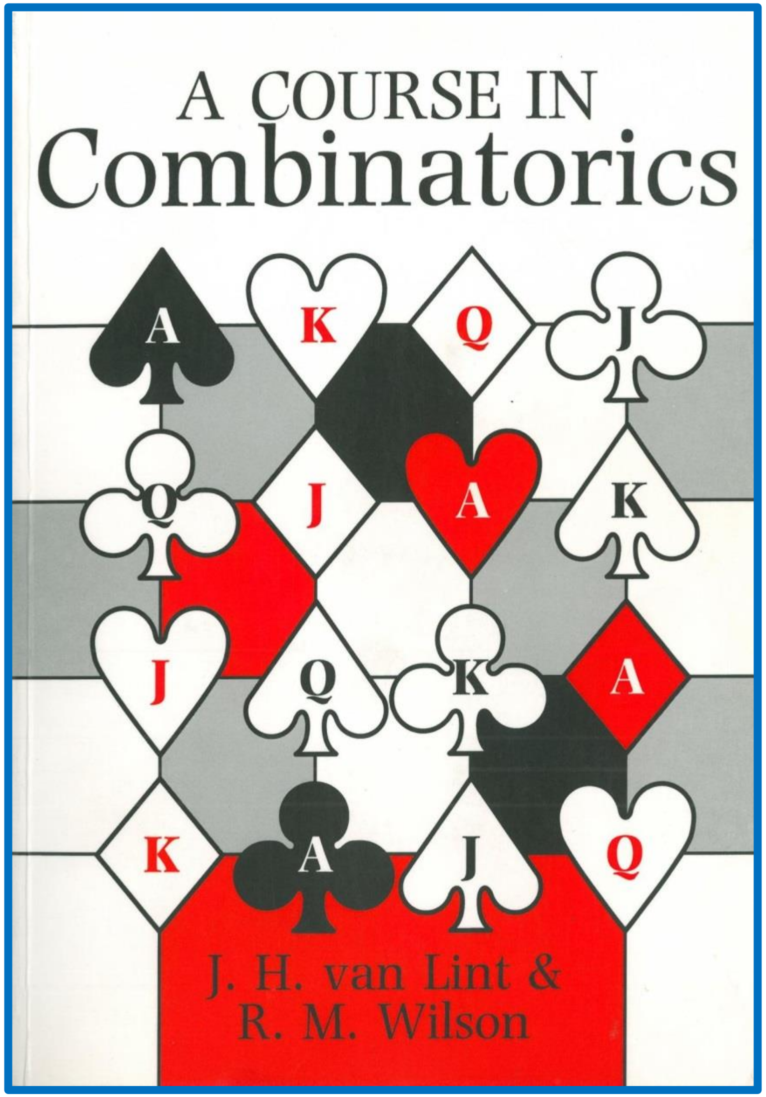
The values (J, Q, K, A) form a latin square – and so do the suits

K♣	Q♦	J♠	A♥
J♥	A♠	K♦	Q♣
A♦	J♣	Q♥	K♠
Q♠	K♥	A♣	J♦

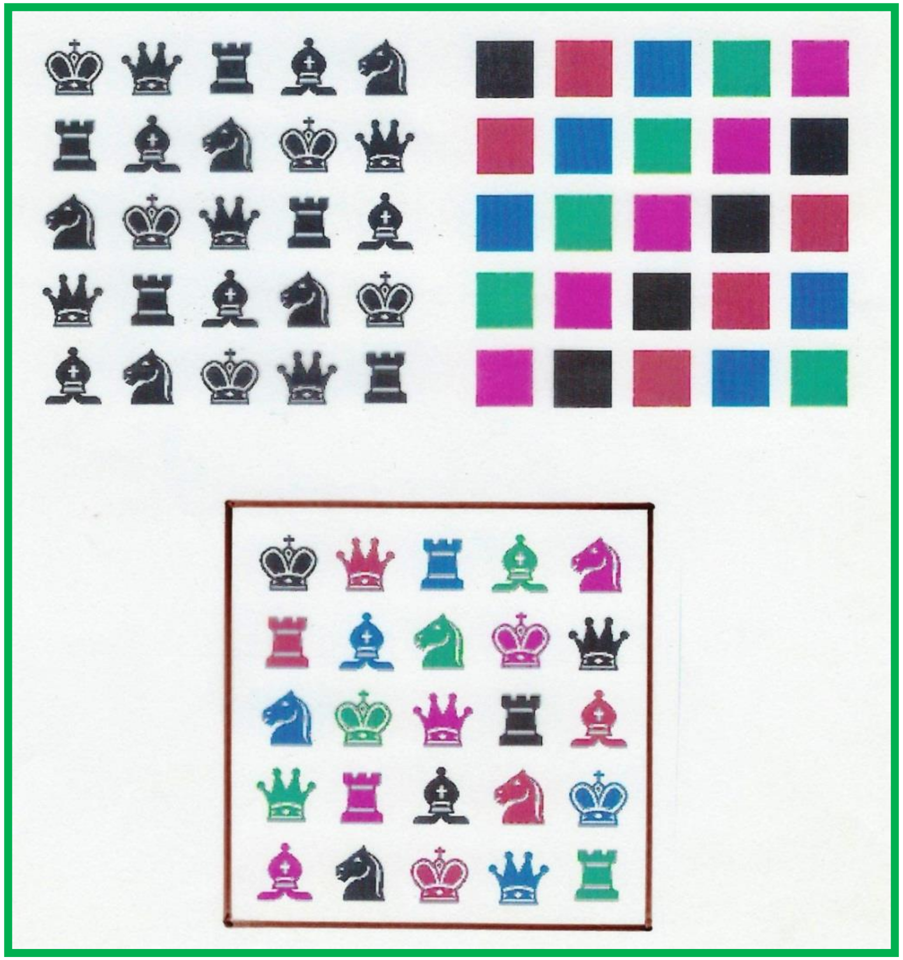
‘Orthogonal 4 × 4 latin squares’

The 16-card problem

AS DE CŒUR	ROI DE TRÈFLE	DAME DE CARREAU	VALET DE PIQUE
VALET DE CARREAU	DAME DE PIQUE	ROI DE CŒUR	AS DE TRÈFLE
ROI DE PIQUE	AS DE CARREAU	VALET DE TRÈFLE	DAME DE CŒUR
DAME DE TRÈFLE	VALET DE CŒUR	AS DE PIQUE	ROI DE CARREAU



Orthogonal 5 × 5 latin squares



Aa	Bb	Cc	Dd	Ee
Cb	Dc	Ed	Ae	Ba
Ec	Ad	Be	Ca	Db
Bd	Ce	Da	Eb	Ac
De	Ea	Ab	Bc	Cd

Each chess-piece and colour appear together just once

Each capital and small letter appear together just once

The first 'Latin square' (Euler)

1^1	2^6	3^4	4^3	5^7	6^5	7^2
2^2	3^7	1^5	5^4	4^1	7^6	6^3
3^3	6^1	5^6	7^5	1^2	4^7	2^4
4^4	5^2	6^7	1^6	7^3	2^1	3^5
5^5	1^3	7^1	2^7	6^4	3^2	4^6
6^6	7^4	4^2	3^1	2^5	5^3	1^7
7^7	4^5	2^3	6^2	3^6	1^4	5^1

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	6	5	7	1	4	2
4	5	6	1	7	2	3
5	1	7	2	6	3	4
6	7	4	3	2	5	1
7	4	2	6	3	1	5

Joseph Sauveur’s solution (1710)

	0	1	2	3	4	5	6
0.0.0.	*Apπ*	*Bqρ*	*Crσ*	*Dſτ*	*Etυ*	*Fuψ*	*Gxχ*
2.3.4.	*Cſυ*	*Dtψ*	*Euχ*	*Fxπ*	*Gpρ*	*Aqσ*	*Brτ*
4.6.1.	*Exρ*	*Fpσ*	*Gqτ*	*Arυ*	*Bſψ*	*Ctχ*	*Duπ*
6.2.1.	*Grψ*	*Aſχ*	*Btπ*	*Cuρ*	*Dxσ*	*Epτ*	*Fqυ*
1.5.2.	*Buσ*	*Cxτ*	*Dpυ*	*Eqψ*	*Frχ*	*Gſπ*	*Atρ*
3.1.6.	*Dqχ*	*Erπ*	*Fſρ*	*Gtσ*	*Auτ*	*Bxυ*	*Cpψ*
5.4.3.	*Ftτ*	*Guυ*	*Axψ*	*Bpχ*	*Cqπ*	*Brρ*	*Eſσ*

Propoſons-nous un Quarré magique de 7 par lettres generales à cõſtruire avec 3 ſortes de lettres *A B C D E F G*: *p q r ſ t u x*: π ρ σ τ υ ψ χ.

Euler's *36 Officers Problem:* 1782

Arrange 36 officers, one of each of six ranks and one of each of six regiments, in a 6 × 6 square array, so that each row and each column contains exactly one officer of each rank and exactly one of each regiment.

1. Une question fort curieuse, qui a exercé pendant quelque temps la sagacité de bien du monde, m'a engagé à faire les recherches suivantes, qui semblent ouvrir une nouvelle carrière dans l'Analyse et en particulier dans la doctrine des combinaisons. Cette question rouloit sur une assemblée de 36 officiers, de six différens grades et tirés de six régimens différens, qu'il s'agissoit de ranger dans un quarré de manière que sur chaque ligne, tant horizontale que verticale, il se trouvât six officiers tant de différens caractères que de régimens différens. Or, après toutes les peines qu'on s'est données pour résoudre ce problème, on a été obligé de reconnoître qu'un tel arrangement est absolument impossible, quoiqu'on ne puisse pas en donner de démonstration rigoureuse.

Is there a pair of orthogonal 6 x 6 latin squares?

Euler's Conjecture

Observing that one can easily construct orthogonal Latin squares of sizes

3×3, 4×4, 5×5 and 7×7,

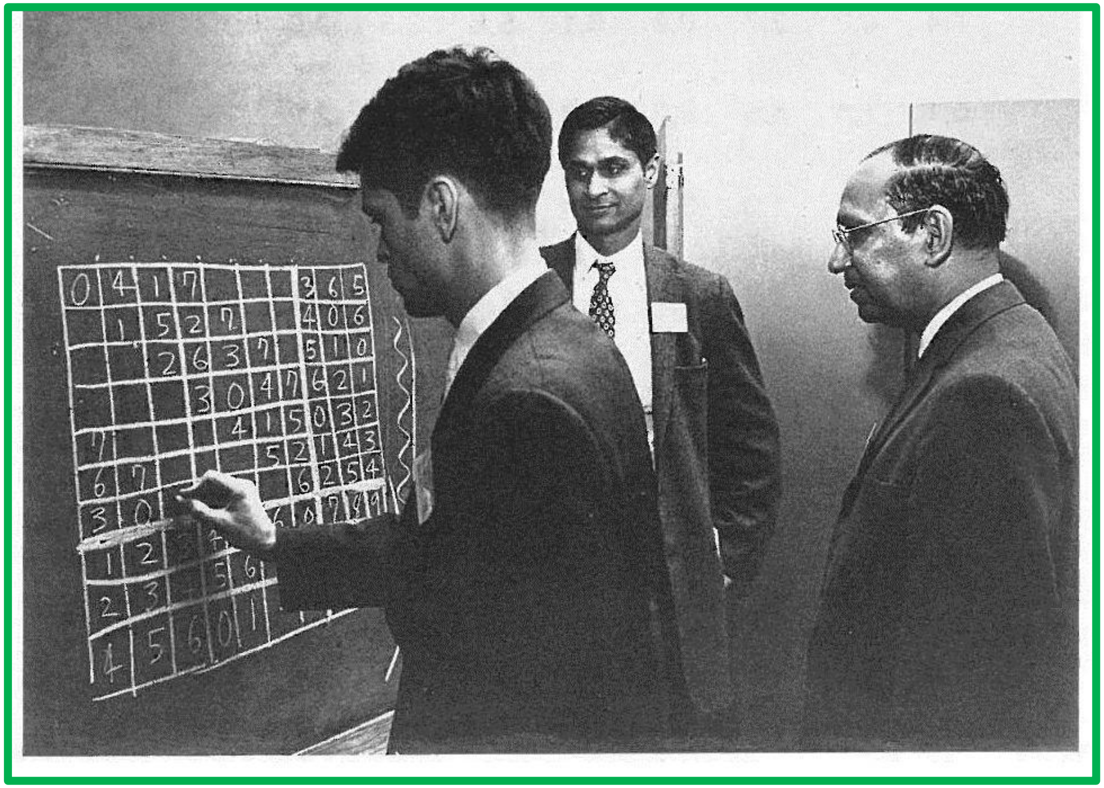
and unable to solve the 36 Officers Problem, Euler conjectured:

Constructing orthogonal $n \times n$ Latin squares is impossible when

$n = 6, 10, 14, 18, 22, \ldots,$

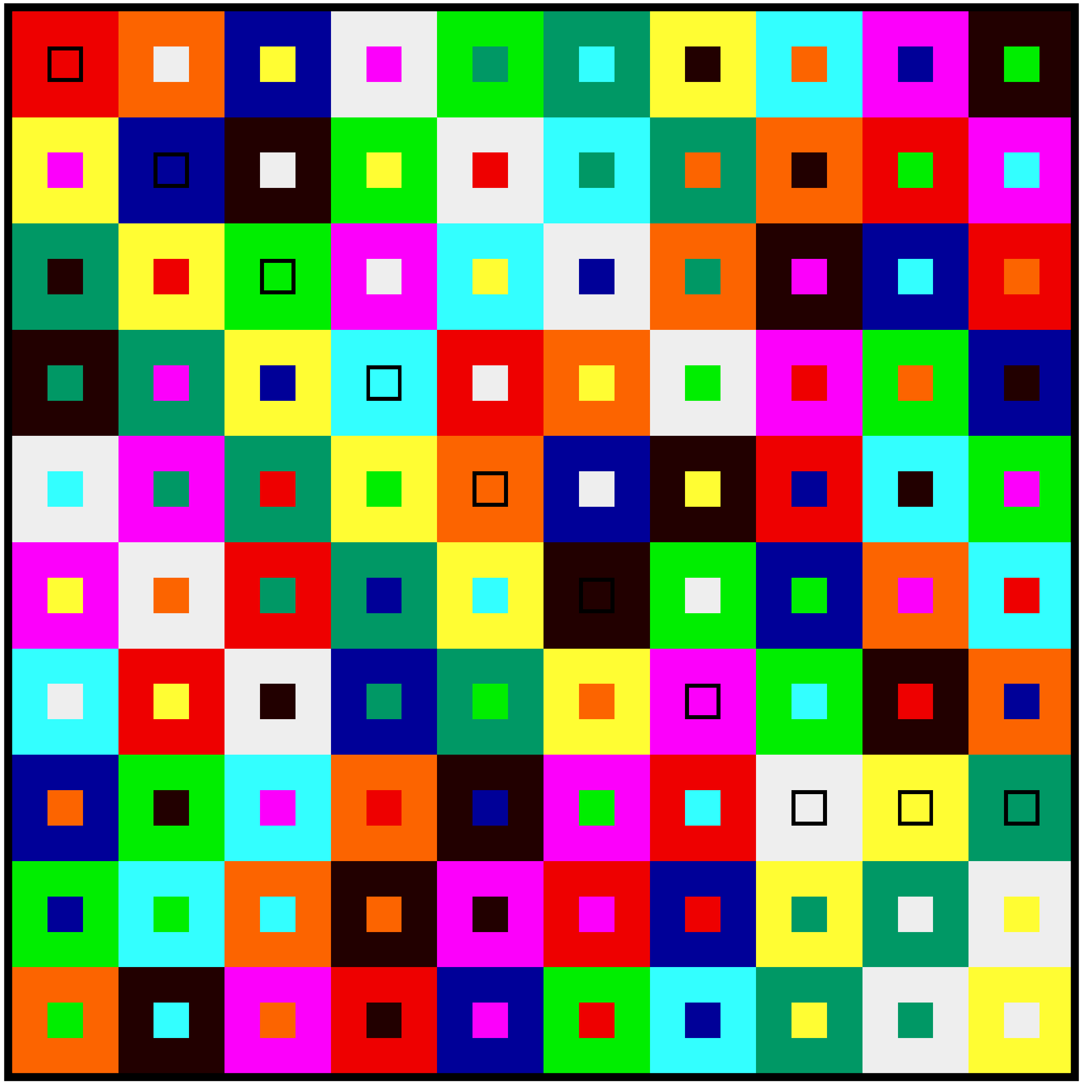
but can be done in all other cases.

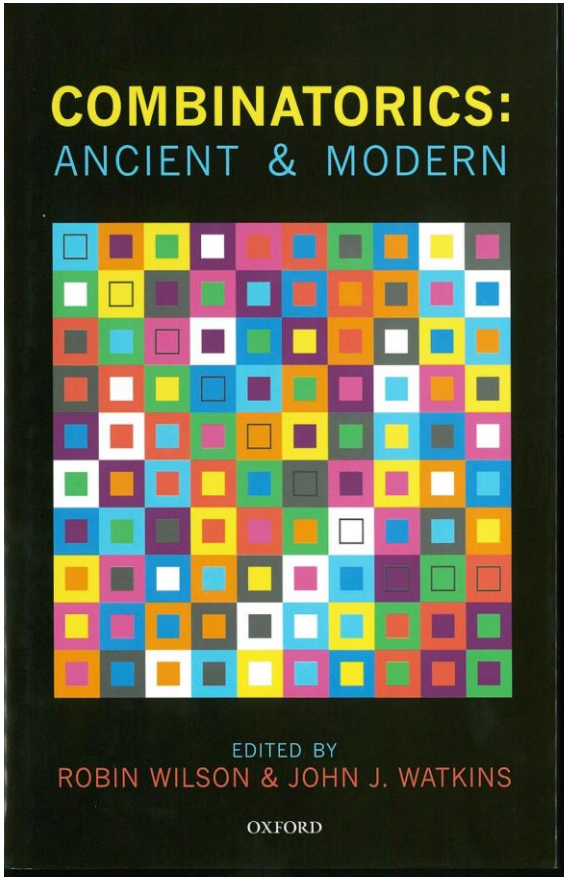
Euler was wrong!



In 1958–60, R. C. Bose, S. Shrikhande and E. T. Parker ('Euler's spoilers') showed that orthogonal latin squares exist for *all* of these values of *n*, except for *n* = 6.

Orthogonal 10 × 10 latin squares





Euler (1782): Converting orthogonal Latin squares to magic squares

a γ	b β	c α
b α	c γ	a β
c β	a α	b γ

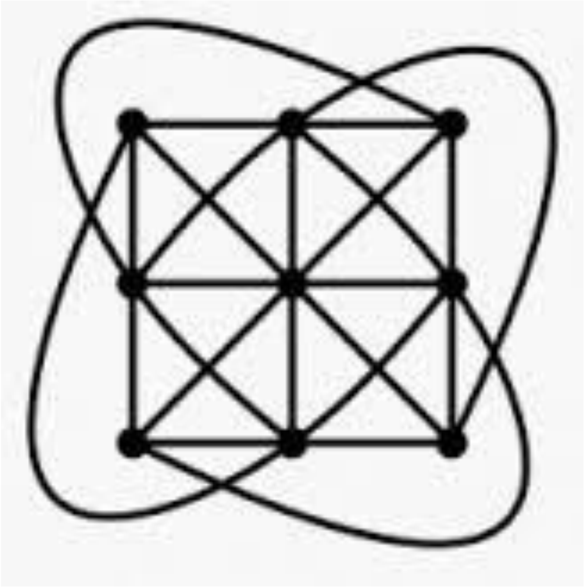
Take a = 0, c = 3, b = 6, and α = 1, γ = 3, β = 3, and add:

2	9	4
7	5	3
6	1	8

Julius Plücker (1835)

A general plane curve has 9 points of inflection, which lie in triples on 12 lines.

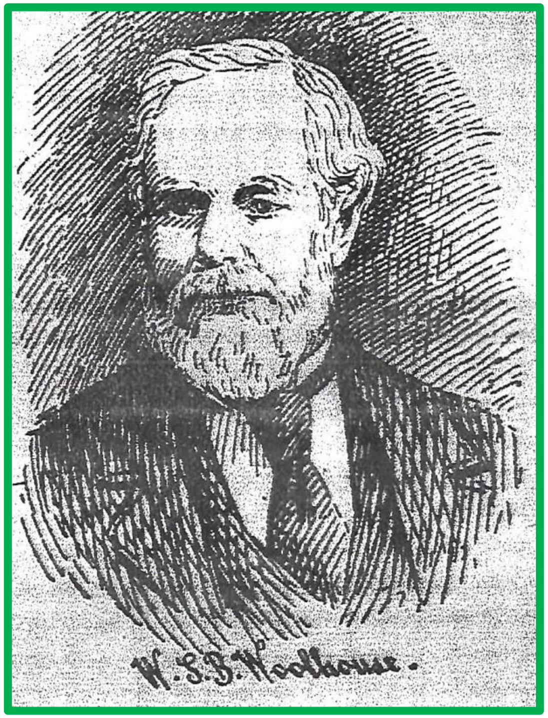
Given any two of the points, exactly one of the lines passes through them both.



Footnote: If a system S(n) of n points can be arranged in triples, so that any two points line in just one triple, then n ≡ 3 (mod 6)

[Later (1839): . . . or n ≡ 1 (mod 6)]

Wesley Woolhouse (1809-93)



Lady's & Gentleman's Diary, 1844

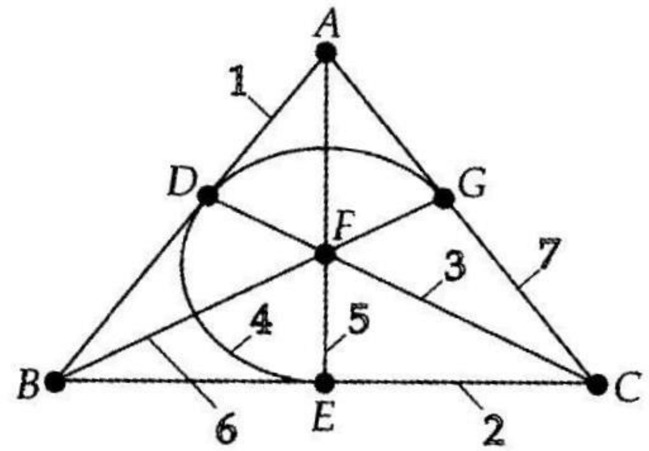
Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other.

Question 1760 (1846):

How many triads can be made out of n symbols, so that no pair of symbols shall be comprised more than once among them? [$p = 3$, $q = 2$]

Triple systems

1	2	3	4	5	6	7
A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C



There are n (= 7) letters, arranged in threes.

Each letter appears in the same number of triples (here, 3).

Any two letters appear together in just one triple.

1	2	3	4	5	6	7	8	9	10	11	12
A	A	A	A	B	B	B	C	C	C	D	G
B	D	E	F	D	E	F	D	E	F	E	H
C	G	I	H	I	H	G	H	G	I	F	I

No. of triples = $n(n-1)/6$, so $n \equiv 1$ or $3 \pmod{6}$

so $n = 7, 9, 13, 15, 19, 21, \ldots$

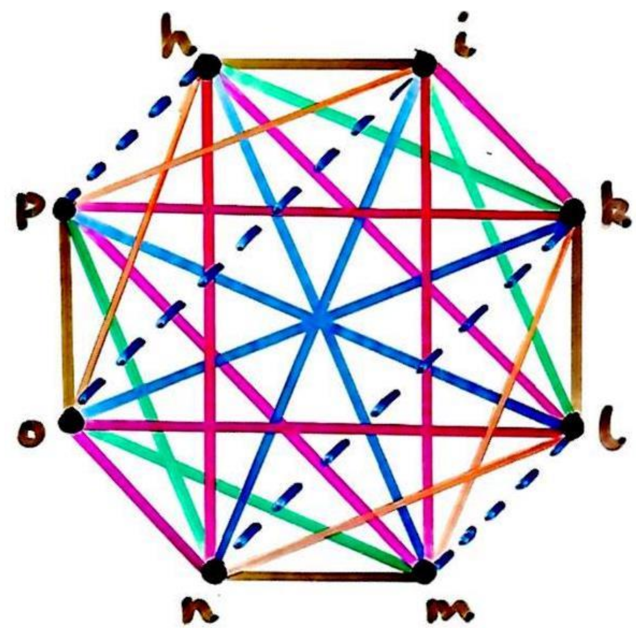
Kirkman's 1847 paper

Cambridge & Dublin Math. J. 2 (1847), 191-204.

Thomas P. Kirkman showed how to construct a triple system S(n) for each n = 1 or 3 (mod 6).

He used a system D_{2m}, an arrangement of the C(2m,2) pairs of 2m symbols in 2m – 1 columns:

hi	hk	hl	hm	hn	ho	hp
kl	il	ik	in	im	ip	io
mn	mo	mp	ko	kp	mk	nk
op	np	no	lp	lo	nl	ml



$S(n), D_{n+1} \rightarrow S(2n + 1)$,

so $S(7), D_8 \rightarrow S(15), D_{16} \rightarrow S(31)$, . . .

Resolvable triple systems: n ≡ 3 (mod 6)

1	2	3	4	5	6	7	8	9	10	11	12
A	A	A	A	B	B	B	C	C	C	D	G
B	D	E	F	D	E	F	D	E	F	E	H
C	G	I	H	I	H	G	H	G	I	F	I

1	11	12	2	6	10	3	7	8	4	5	9
A	D	G	A	B	C	A	B	C	A	B	C
B	E	H	D	E	F	E	F	D	F	D	E
C	F	I	G	H	I	I	G	H	H	I	G

Nine young ladies in a school walk out three abreast for four days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

Steiner triple systems? (1853)

11.

Combinatorische Aufgabe.

(Von Herrn Professor Dr. *J. Steiner* zu Berlin.)

---

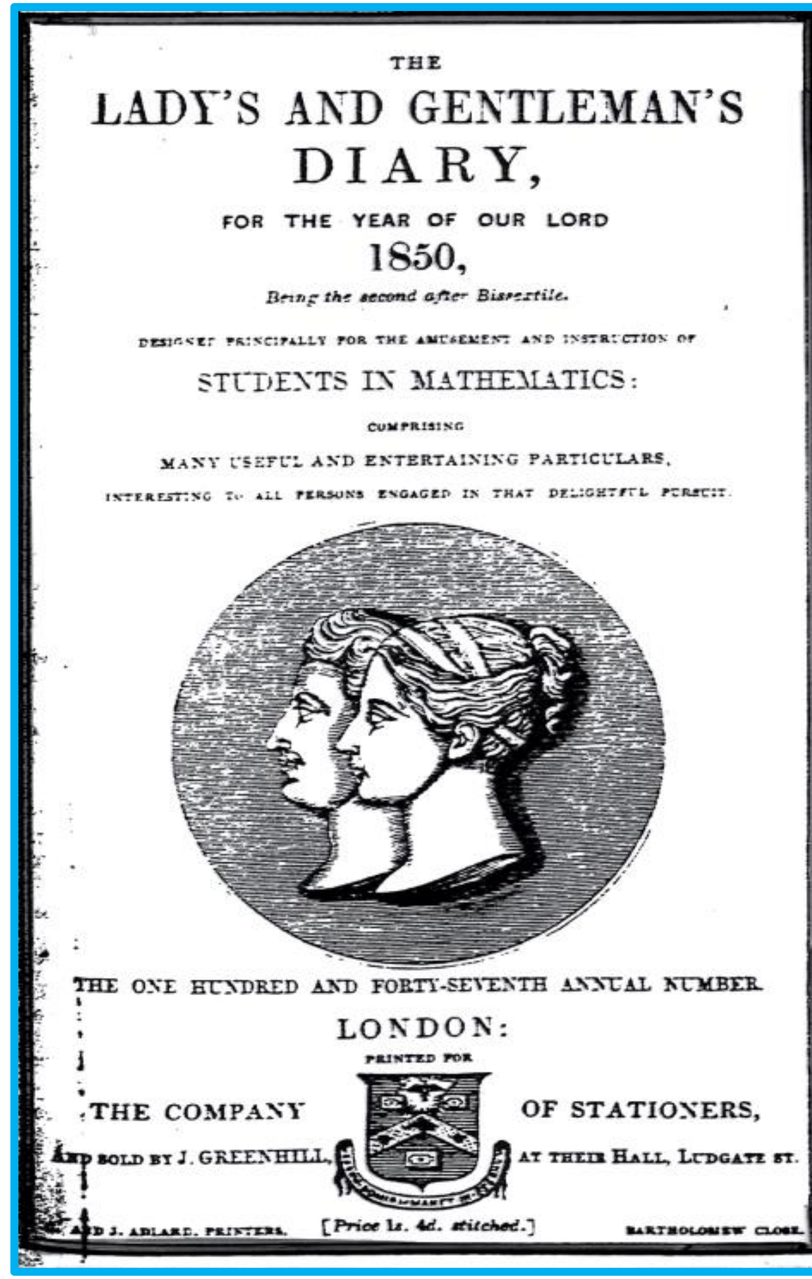
a) Welche Zahl, N, von Elementen hat die Eigenschaft, dafs sich die Elemente so zu dreien ordnen lassen, dafs je zwei in *einer*, aber *nur in einer* Verbindung vorkommen? Wie viele wesentlich verschiedene Anordnungen, d. h. solche, die nicht durch eine blofse Permutation der Elemente auseinander hervorgehen, giebt es bei jeder Zahl?

b) Wenn ferner die Elemente sich so zu vieren verbinden lassen sollen, dafs jede drei freien Elemente, d. h. solche, welche nicht schon einen der vorigen Dreier (*a*.) bilden, immer in *einem* aber *nur in einem* Vierer vorkommen, und dafs auch keine 3 Elemente eines solchen Vierers einem der vorigen Dreier angehören; so entsteht daraus keine neue Bedingung für die Zahl N.

aufgefundenen Form, die geforderten Verbindungen auch in der That möglich sind. — Wenn z. B. in Rücksicht der ersten Bedingung (*a*.) allein die Zahl N von der Form $6n+1$ oder $6n+3$ sein mufs, so ist zu beweisen, dafs für jede Zahl von einer dieser zwei Formen auch in der That die N Elemente sich auf die geforderte Art zu $\frac{1}{6}N(N-1)$ Dreiern verbinden lassen. Nämlich aus den gestellten Bedingungen folgt leicht, dafs

$$\text{die Zahl der Dreier} = \frac{N(N-1)}{2.3},$$

Lady's and Gentleman's Diary (1850)



THE

LADY'S AND GENTLEMAN'S

DIARY,

FOR THE YEAR OF OUR LORD

1850,

Being the second after Bissextile.

DESIGNED PRINCIPALLY FOR THE AMUSEMENT AND INSTRUCTION OF

STUDENTS IN MATHEMATICS:

COMPRISING

MANY USEFUL AND ENTERTAINING PARTICULARS,

INTERESTING TO ALL PERSONS ENGAGED IN THAT DELIGHTFUL PURSUIT.

THE ONE HUNDRED AND FORTY-SEVENTH ANNUAL NUMBER.

LONDON:

PRINTED FOR

THE COMPANY OF STATIONERS,

AND SOLD BY J. GREENHILL, AT THEIR HALL, LUDGATE ST.

AND J. ADLARD, PRINTERS. [*Price 1s. 4d. stitched.*] BARTHOLOMEW CLOSE.

Lady’s & Gentleman’s Diary, 1850

I. QUERY; *by* MR JAMES LUGG, *Grampound, Cornwall.*

Required the origin of the custom of making fools on the first day of April?

II. QUERY; *by the* REV. JOHN HOPE, *Stapleton.*

Does there seem to be anything prophetic in the names of the three sons of Noah, *Shem*, *Ham*, and *Japheth*?

III. QUERY; *by* MR. JOHN ELLIOTT, *of Stanhope.*

What is the cause of the contraction of hemp and catgut strings in a damp atmosphere?

IV. QUERY; *by* MR. JAMES HERDSON, *Tobermory.*

How is the saltness of the sea accounted for? And does the saltness increase or not?

V. QUERY; *by* MR. THOMAS MARTIN, *Birmingham.*

Was the origin of the National Anthem, “God save the Queen,” in any way connected with the Diary?

VI. QUERY: *by the* REV. THOS. P. KIRKMAN, *Croft, near Warrington.*

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

PRINTED FOR THE COMPANY OF STATIONERS.

Solving the 'schoolgirls problem'

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

Monday:	A–B–C	D–E–F	G–H–I	J–K–L	M–N–O
Tuesday:	A–D–G	B–E–H	C–L–O	F–J–N	I–K–M
Wednesday:	A–J–M	B–K–N	C–F–I	D–H–L	E–G–O
Thursday:	A–E–K	B–F–L	C–G–M	D–I–N	H–J–O
Friday:	A–H–N	B–I–O	C–D–J	F–G–K	E–L–M
Saturday:	A–F–O	B–D–M	C–H–K	E–I–J	G–L–N
Sunday:	A–I–L	B–G–J	C–E–N	F–H–M	D–K–O

Kirkman's problem of the fifteen young ladies

A governess of great repute,
Young ladies had *fifteen*,
Who took their walks along the shore,
Or in the meadows green.

But as they walked they tattled and talked
In chosen *groups of three*,
Until their governess resolved,
Such trifling should not be.

For she would try for *one whole week*,
So to arrange them all,
That *no two girls a second time*
In the same rank should fall.

Kirkman's 'schoolgirls problem'

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

Answered by the Rev. Mr. Kirkman, *the Proposer.*

Denoting the ladies by a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3; e_1, e_2, e_3, the following arrangement will be found to answer the question:

$a_1\ a_2\ a_3$	$a_1\ b_1\ c_1$	$a_1\ d_1\ e_1$	$a_1\ b_2\ d_2$	$a_1\ c_2\ e_2$	$a_1\ b_3\ e_3$	$a_1\ c_3\ d_3$
$b_1\ b_2\ b_3$	$a_2\ b_2\ c_2$	$a_2\ d_2\ e_2$	$a_2\ b_3\ d_3$	$a_2\ c_3\ e_3$	$a_2\ b_1\ e_1$	$a_2\ c_1\ d_1$
$c_1\ c_2\ c_3$	$a_3\ d_3\ e_3$	$a_3\ b_3\ c_3$	$a_3\ c_1\ e_1$	$a_3\ b_1\ d_1$	$a_3\ c_2\ d_2$	$a_3\ b_2\ e_2$
$d_1\ d_2\ d_3$	$b_3\ d_1\ e_2$	$d_3\ b_1\ c_2$	$b_1\ c_3\ e_2$	$c_1\ b_3\ d_2$	$b_2\ c_3\ d_1$	$c_2\ b_3\ e_1$
$e_1\ e_2\ e_3$	$c_3\ d_2\ e_1$	$e_3\ b_2\ c_1$	$d_1\ c_2\ e_3$	$e_1\ b_2\ d_3$	$e_2\ c_1\ d_3$	$d_2\ b_1\ e_3$

This is the symmetrical and only possible solution. All others differ from this only in disturbing the alphabetical order, or that of the three subindices in certain triplets of the first column, or in both these together.

Again by Mr. Samuel Bills, *Hawton, near Newark-upon-Trent;* Mr. Thomas Jones, *Abbey Buildings, Chester:* Mr. Thomas Wainman, *Burley, near Leeds; and* Mr. W. H. Levy, *Shalbourne, near Hungerford.*

Suppose the fifteen young ladies to be distinguished by the numerals 1, 2, 3—15. They may be arranged in the following way for the seven days:

1st Day.	2d Day.	3d Day.	4th Day.	5th Day.	6th Day.	7th Day.
1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
4 8 12	2 8 10	2 12 14	2 13 15	2 4 6	2 5 7	2 9 11
5 11 14	3 12 15	3 8 11	3 5 6	3 13 14	3 9 10	3 4 7
6 9 15	6 11 13	4 9 13	4 10 14	5 9 12	4 11 15	5 8 13
7 10 13	7 9 14	5 10 15	7 11 12	7 8 15	6 8 14	6 10 12

In the above arrangement no two of the young ladies walk twice abreast.

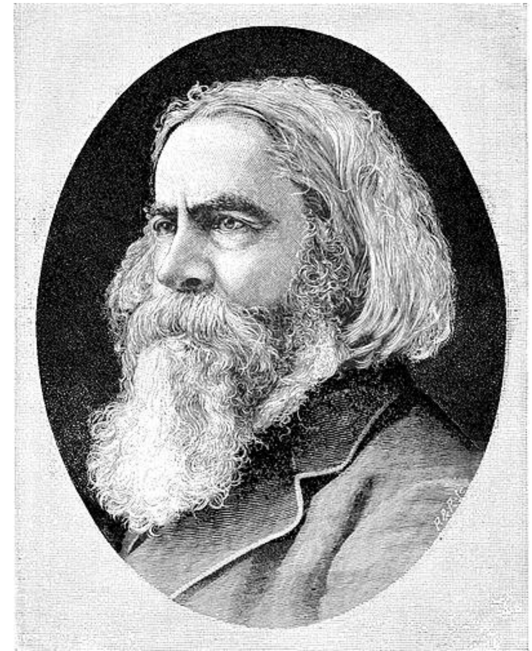
Cyclic solutions

Revd. Robert Anstice (1852)

0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, ∞

∞ 0 0	1 2 3	1 4 5	3 5 6	4 2 6
∞ 1 1	2 3 4	2 5 6	4 6 0	5 3 0
∞ 2 2	3 4 5	3 6 0	5 0 1	6 4 1
∞ 3 3	4 5 6	4 0 1	6 1 2	0 5 2
∞ 4 4	5 6 0	5 1 2	0 2 3	1 6 3
∞ 5 5	6 0 1	6 2 3	1 3 4	2 0 4
∞ 6 6	0 1 2	0 3 4	2 4 5	3 1 5

Benjamin Peirce
(1860)



Using the same approach as Anstice, Peirce found all three types of cyclic solution:

∞ 0 0	1 2 3	1 4 5	3 5 6	4 2 6	[Anstice]
∞ 0 0	1 3 4	2 4 5	3 5 6	1 2 6	[Cayley]
∞ 0 0	1 5 6	3 4 6	1 2 4	3 2 5	[Kirkman]

The seven schoolgirls problem solutions

F. N. Cole, Bull. AMS (1922)

I–II

	M	Tu	W	Th	F	Sat	Sun
K	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	1 4 7 2 5 8 3 10 13 6 11 14 9 12 15	1 5 15 2 9 10 3 4 14 6 8 12 7 11 13	1 9 13 2 4 12 3 5 11 6 7 15 8 10 14	1 6 10 2 11 15 3 7 12 4 8 13 5 9 14	1 8 11 2 7 14 3 6 9 4 10 15 5 12 13	1 12 14 2 6 13 3 8 15 4 9 11 5 7 10
C	"	"	"	"	1 6 10 2 7 14 3 8 15 4 9 11 5 12 13	1 8 11 2 6 13 3 7 12 4 10 15 5 9 14	1 12 14 2 11 15 3 6 9 4 8 13 5 7 10

III–IV

	M	Tu	W	Th	F	Sat	Sun
3	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	1 4 7 2 5 8 3 10 13 6 11 14 9 12 15	1 5 15 2 9 13 3 4 11 6 7 12 8 10 14	1 9 10 2 4 12 3 5 14 6 8 15 7 11 13	1 6 13 2 7 14 3 8 12 4 10 15 5 9 11	1 8 11 2 6 10 3 7 15 4 9 14 5 12 13	1 12 14 2 11 15 3 6 9 4 8 13 5 7 10
4	"	"	"	"	1 6 13 2 11 15 3 8 12 4 9 14 5 7 10	1 12 14 2 6 10 3 7 15 4 8 13 5 9 11	1 8 11 2 7 14 3 6 9 4 10 15 5 12 13

V–VI

	M	Tu	W	Th	F	Sat	Sun
5	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	1 4 7 2 5 10 3 8 13 6 11 14 9 12 15	1 5 13 2 4 12 3 9 11 6 7 15 8 10 14	1 6 8 2 11 13 3 7 12 4 10 15 5 9 14	1 9 10 2 7 14 3 5 15 4 8 11 6 12 13	1 11 15 2 6 9 3 4 14 5 8 12 7 10 13	1 12 14 2 8 15 3 6 10 4 9 13 5 7 11
6	"	"	"	"	1 9 10 2 8 15 3 4 14 5 7 11 6 12 13	1 11 15 2 7 14 3 6 10 4 9 13 5 8 12	1 12 14 2 6 9 3 5 15 4 8 11 7 10 13

VII

	M	Tu	W	Th	F	Sat	Sun
7	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	1 4 7 2 5 10 3 8 13 6 11 14 9 12 15	1 5 9 2 4 14 3 10 15 6 8 12 7 11 13	1 6 15 2 9 11 3 7 14 4 8 10 5 12 13	1 8 11 2 6 13 3 4 12 5 7 15 9 10 14	1 10 13 2 7 12 3 6 9 4 11 15 5 8 14	1 12 14 2 8 15 3 5 11 6 7 10 4 9 13

A priority dispute?

J. J. Sylvester, *Phil. Mag.* 1861

I may also take occasion to observe that, in connexion with my researches in combinatorial aggregation, long before the publication of my unfinished paper in the Magazine *[1844] I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all duads to the base 15, which has since become so well known, and fluttered so many a gentle bosom, under the title of the fifteen school-girls' problem; and it is not improbable that the question, under its existing form, may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge long years before its first appearance, which I believe was in the* Ladies' Diary *for some year which my memory is unable to furnish.*

Kirkman replies . . .

My distinguished friend Professor Sylvester ... volunteers en passant *an hypothesis as to the possible origin of this noted puzzle under its existing form. No man can doubt, after reading his words, that he was in possession of the property in question of the number 15 when he was an Undergraduate at Cambridge.*

But the difficulty of tracing the origin of the puzzle, from my own brains to the fountain named at that University, is considerably enhanced by the fact that, when I proposed the question in 1849, I had never had the pleasure of seeing either Cambridge or Professor Sylvester.

My own account of the origin of the problem may be seen at p. 260, vol. v., of the Cambridge and Dublin Mathematical Journal*, 1850. No other account of it has, so far as I know, been published in print except this guess of Prof. Sylvester's in 1861.*

Sylvester's problem

There are C(15, 3) = 455 = 13 x 35 triples of schoolgirls.
Are there 13 separate solutions that use all 455 triples?
-- that is, can we arrange 13 weekly schedules so that each triple appears just once in the quarter-year?

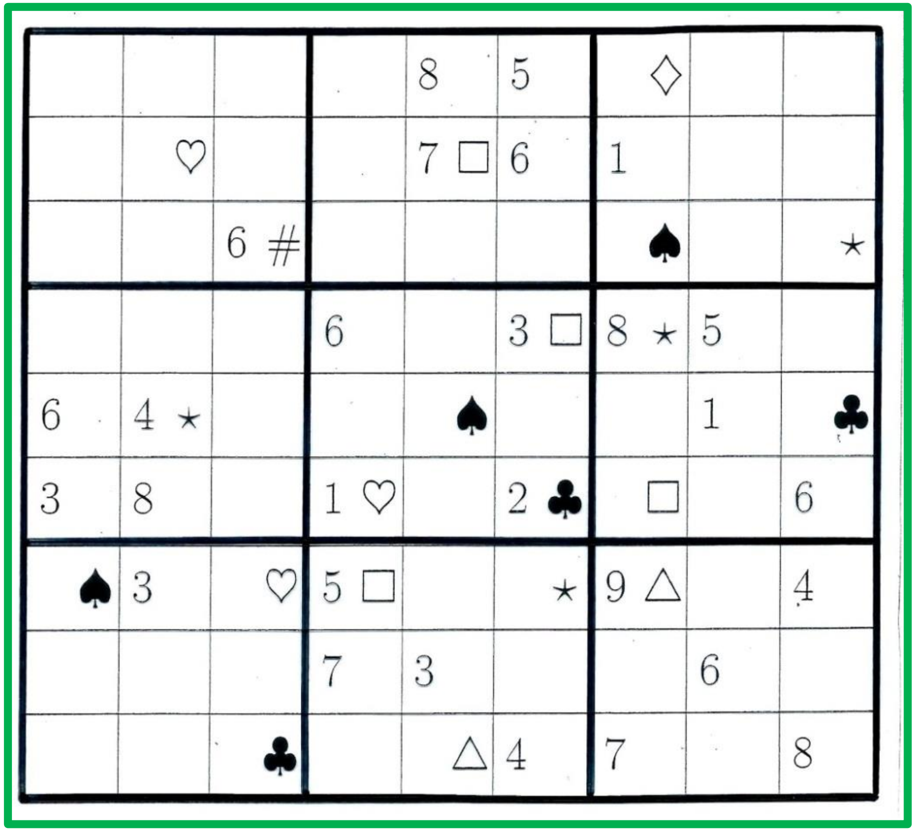
Yes? – Kirkman (1850) – but his solution was incorrect.

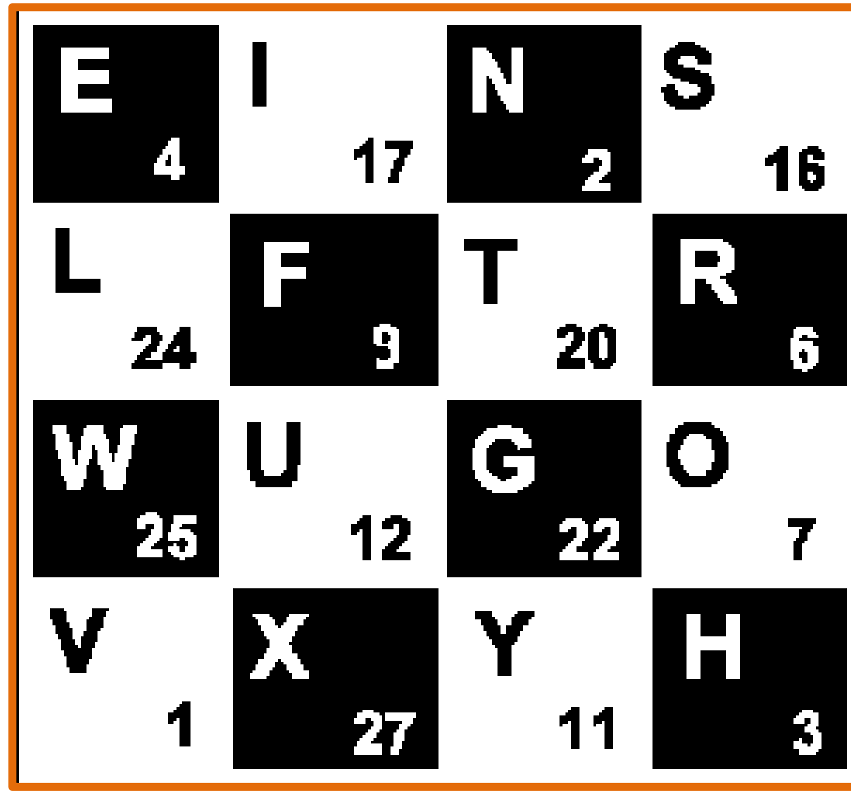
Yes: R. H. F. Denniston (using a computer) in 1974.

A solution of the schoolgirls problem for n = 6k + 3 schoolgirls was given in 1971 by Dijen Ray-Chaudhuri and Rick Wilson (and had been found earlier by Lu Xia Xi, a schoolteacher from Inner Mongolia).

The solution of the generalized Sylvester problem for n = 6k + 3 schoolgirls is still unknown.

Two puzzles


8 5 ◇
♡ 7 □ 6 1
6 # ♠ ⋆
6 3 □ 8 ⋆ 5
6 4 ⋆ ♠ 1 ♣
3 8 1 ♡ 2 ♣ □ 6
♠ 3 ♡ 5 □ ⋆ 9 △ 4
7 3 6
♣ △ 4 7 8


E 4
I 17
N 2
S 16
L 24
F 9
T 20
R 6
W 25
U 12
G 22
O 7
V 1
X 27
Y 11
H 3