On finite non-solvable 4-primary groups without elements of order 6

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This is joint work with Anatoly Kondrat'ev

We use mainly standard notation and terminology (see [1]).

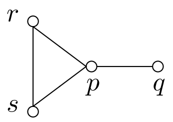
Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. If $|\pi(G)| = n$, then G is called n-primary. The Gruenberg–Kegel graph (prime graph) $\Gamma(G)$ of G is a graph with the vertex set $\pi(G)$, in which two distinct vertices p and q are adjacent if and only if there exists an element of order pq in G.

In 2012–2013, A.S. Kondrat'ev described finite groups having the same Gruenberg-Kegel graph as the groups $Aut(J_2)$ [2] and A_{10} [3], respectively. The Gruenberg–Kegel graphs of these groups are isomorphic as abstract graphs.

We establish a more general problem: to describe finite groups whose Gruenberg-Kegel graphs are isomorphic as abstract graphs to the graph $\Gamma(A_{10})$.

As a part of the solution of this problem, we proved in [4] that if G is a finite non-solvable group and the graph $\Gamma(G)$ as abstract graph is isomorphic to the graph $\Gamma(A_{10})$, then the quotient group $G/S(G)$ (where $S(G)$ is the solvable radical of G) is almost simple, and classified all finite almost simple groups whose the Gruenberg-Kegel graphs as abstract graphs are isomorphic to subgraphs of $\Gamma(A_{10})$.

Let G be a finite non-solvable group, and the graph $\Gamma(G)$ as abstract graph is isomorphic to the graph $\Gamma(A_{10})$. Then the graph $\Gamma(G)$ has the following form



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where p, q, r, and s are pairwise distinct primes.

In [5], we described such finite non-solvable groups G when 3 does not divide $|G|$.

In this work, we prove the following theorem.

Theorem. *Let G be a finite non-solvable group, $\overline{G} \cong G/S(G)$ and $\Gamma(G)$ as abstract graph is isomorphic to $\Gamma(A_{10})$. If 3 divides of $|G|$ and G has no elements of order 6, then one of the following statements holds:*

(1) *$S = O_{2',2}(G)$, $O(G) = O_p(G)$, $q = 2$, $S/O(G)$ is an elementary abelian 2-group, $\overline{G} \cong L_2(2^n)$ and one of the following statements holds:*

(1*a*) *$n = 4$, $p = 17$ and $\{r, s\} = \{3, 5\}$;*

(1*b*) *n is prime, $n \geq 5$, $p = 2^n - 1$, and $\{r, s\} = \{3, (2^n - 1)/3\}$, the group $S/O(G)$ either is trivial or as $\overline{G}$-module is isomorphic to a direct sum of natural $GF(2^n)\overline{G}$-modules;*

(2) *$S = O_p(G)$, $q = 2$, $\overline{G} \cong L_2(p)$, $p \geq 31$, $p \equiv \varepsilon 5 (mod\ 12)$, $\varepsilon \in \{+, -\}$, $p - \varepsilon 1 = 2^k$, and $3 \in \{r, s\} = \pi((t + \varepsilon 1)/2)$;*

(3) *$S = O_p(G)$, $q = 3$, and one of the following statements holds:*

(3*a*) *$\overline{G} \cong PGL_2(9)$, $p > 5$, and $\{r, s\} = \{2, 5\}$;*

(3*b*) *$\overline{G} \cong L_2(81)$, $PGL_2(81)$ or $L_2(81).2_3$, $p = 41$, and $\{r, s\} = \{2, 5\}$;*

(3*c*) *$\overline{G} \cong L_2(3^n)$ or $PGL_2(3^n)$, n is an odd prime, $p = (3^n - 1)/2$, and $\{r, s\} = \pi(3^n + 1)$.*

Each of the statements (1)–(3) of the theorem is realized.

In the proof of Theorem, we use the classification of the finite non-solvable groups without elements of order 6 from [6, Theorem 2].

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References

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