On the pronormality of subgroups of odd index in some direct products of finite groups

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We consider only finite groups. A subgroup H of a group G is said to be *pronormal* in G if H and H^g are conjugate in $\langle H, H^g\rangle$ for each $g \in G$. Some of well-known examples of pronormal subgroups are the following: normal subgroups; maximal subgroups; Sylow subgroups; Sylow subgroups of proper normal subgroups; Hall subgroups of solvable groups. Some problems in finite group theory, combinatorics, and permutation group theory were solved in terms of the pronormality, see, for example [1, 5]. Thus, the problem of pronormality of a subgroup (in particular, of a subgroup of odd index) in a finite group is of interest. Detailed surveys of investigations on pronormality of subgroups of odd index in finite groups could be found in survey papers [3, 4]. These surveys contain new results and some conjectures and open problems. One such open problem is to complete the classification of finite simple groups in which all the subgroups of odd index are pronormal. One more open problem involves the classification of direct products of nonabelian simple groups in which the subgroups of odd index are pronormal. A detailed motivation for these problems was provided in [2].

In this talk we concentrate on the latter problem. Note that there are examples of nonabelian simple groups G such that all the subgroups of odd index are pronormal in G, but the group $G \times G$ contains a non-pronormal subgroup of odd index (see [2, Proposition 1]). We prove the following theorem.

Theorem. *Let $G = \prod_{i=1}^{t} G_i$, where for each $i \in \{1, ..., t\}$, $G_i \cong Sp_{n_i}(q_i)$ for odd q_i. Then the following statements are equivalent*:

(1) *all the subgroups of odd index are pronormal in G*;
(2) *for each i, all the subgroups of odd index are pronormal in G_i*;
(3) *for each i, if $q_i \equiv \pm 3 \pmod 8$, then n_i is either a power of 2 or is a number of the form $2^w(2^{2k}+1)$.*

Moreover, guided by the analysis in the proof of Theorem, we conclude that deciding the pronormality of a given subgroup of odd index in the direct product of simple symplectic groups over fields of odd characteristics is reducible to deciding the pronormality of some subgroup H of odd index in a subgroup of a group $\prod_{i=1}^{t} Z_i \wr Sym_{n_i}$, where $Z_i \in \{1, C_3\}$ for each i, such that H projects onto $\prod_{i=1}^{t} Sym_{n_i}$, and we have obtained a criterium of pronormality of such a subgroup in such a group. These investigations give a rise to researches on effective algorithms (to be implemented in GAP) for deciding the pronormality of a subgroup of odd index in a finite simple symplectic group, the speaker is working on these algorithms in a joint project with Stephen Glasby and Cheryl E. Praeger.

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