On spectral properties of the Star graphs

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The Star graph S_n, $n \geqslant 2$, is the Cayley graph over the symmetric group Sym_n generated by transpositions swapping the ith element of the permutation with the first one. It is a connected bipartite $(n-1)$-regular graph of order $n!$, and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [1].

In 2012, R. Krakovski and B. Mohar [7] proved that the spectrum of S_n contains all integers in the range from $-(n-1)$ up to $n-1$. Since the Star graph is bipartite, the spectrum is symmetric and multiplicities of eigenvalues of $(n-k)$ and $-(n-k)$ are equal for each integer $1 \leqslant k < n$. Furthermore, $\pm(n-1)$ are simple eigenvalues of S_n. At the same time, G. Chapuy and V. Feray [3] showed that the spectrum of the Star graphs is equivalent to the spectrum of Jucys-Murphy elements in the algebra of the symmetric group. This connection between two kinds of spectra implies that the Star graph is integral.

In 2016, S. V. Avgustinovich, E. N. Khomyakova and E. V. Konstantinova [2] suggested a method for getting explicit formulas for multiplicities of eigenvalues $\pm(n-k)$ and presented such formulas for $2 \leqslant k \leqslant 5$. Moreover, an asymptotic lower bound was obtained. It was proved that for a fixed integer eigenvalue of S_n, its multiplicity is at least $2^{\frac{1}{2} n \log n (1-o(1))}$ for sufficiently large n. In 2018, E. N. Khomyakova [6] investigated the behavior of the eigenvalues multiplicity function of S_n for eigenvalues $\pm(n-k)$ where $1 \leqslant k \leqslant \frac{n+1}{2}$. The main result is given by the following theorem.

Theorem 1. *Let $n, k \in \mathbb{Z}$, $n \geqslant 2$ and $1 \leqslant k \leqslant \frac{n+1}{2}$, then the multiplicity* $\mathrm{mul}(n-k)$ *of eigenvalue $(n-k)$ of the Star graph S_n is calculated by the formula:*

$$\mathrm{mul}(n-k) = \frac{n^{2(k-1)}}{(k-1)!} + P(n),$$

where $P(n)$ is a polynomial of degree $2k-3$.

In 2019, E. N. Khomyakova and E. V. Konstantinova [5] presented explicit formulas for calculating multiplicities of eigenvalues $\pm(n-k)$ where $2 \leqslant k \leqslant 12$ and firstly collected computational results of all eigenvalue multiplicities for $n \leqslant 50$ in a catalogue (`https://link.springer.com/article/10.1007/s40065-019-00271-z`). This exact values show that Theorem 1 holds for any $n \geqslant 2$ and $1 \leqslant k \leqslant n$. Authors used computational results to get diagrams with plotting them on a logarithmic scale with base 2 such that the abscissa corresponds to the eigenvalues of the Star graphs S_n for a fixed n and the ordinate corresponds to the multiplicities [4]. In case k is fixed, diagram looks like a polynomial function by Theorem 1. In case n is fixed, diagram in normal scale contains exponential rises and falls appear. Thus the function may be a straight exponent for sufficiently large n, but it is just conjecture.

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References

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