The volume of a general type hyperbolic tetrahedron in terms of its edge matrix

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This is joint work with Bao Vuong

A *compact hyperbolic tetrahedron* T is a convex hull of four points in the hyperbolic space $\mathbb{H}^3$. Let us denote the vertices of T by numbers $1, 2, 3$ and 4. Then denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. We put θ_{ij} for the dihedral angle along the corresponding edge. It is well known that T is uniquely defined up to isometry either by the set of its dihedral angles or the set of its edge lengths. A *Gram matrix* $G(T)$ of tetrahedron T is defined as $G(T) = \langle -\cos\theta_{ij}\rangle_{i,j=1,2,3,4}$, we assume here that $-\cos\theta_{ii} = 1$. An *edge matrix* $E(T)$ of hyperbolic tetrahedron T is defined as $E(T) = \langle\cosh\ell_{ij}\rangle_{i,j=1,2,3,4}$, where $\ell_{ii} = 0$.

In 1907 G. Sforza found the volume of a hyperbolic tetrahedron T in terms of its Gram matrix (see [1]). The new proof of the Sforza's formula was recently given in [2]. In the present work we present an exact formula for the volume of a hyperbolic tetrahedron T in terms of its edge matrix.

Theorem 1. *Let T be a compact hyperbolic tetrahedron given by its edge matrix $E = E(T)$ and $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of E. We assume that all the edge lengths are fixed exept ℓ_{34} which is formal variable. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{2}\int\limits_{f_1}^{\ell_{34}}\Bigg[\frac{t}{(-\det E)^{3/2}}\left(\frac{c_{14}(c_{11}c_{23}-c_{12}c_{13})}{c_{11}}+\frac{c_{24}(c_{13}c_{22}-c_{12}c_{23})}{c_{22}}\right)-$$
$$\frac{\sinh t}{(-\det E)^{1/2}}\left(\frac{c_{14}\ell_{24}\sinh\ell_{24}+c_{13}\ell_{14}\sinh\ell_{23}}{c_{11}}+\frac{c_{23}\ell_{13}\sinh\ell_{13}+c_{24}\ell_{23}\sinh\ell_{14}}{c_{22}}+\ell_{12}\sinh\ell_{12}\right)\Bigg]\,dt,$$

$$\cosh f_1 = \cosh\ell_{13}\cosh\ell_{14} - (\cosh\ell_{13}\cosh\ell_{12}-\cosh\ell_{23})(\cosh\ell_{14}\cosh\ell_{12}-\cosh\ell_{24})\operatorname{csch}^2\ell_{12} -$$
$$\sqrt{(\cosh\ell_{23}-\cosh(\ell_{13}+\ell_{12}))(\cosh\ell_{23}-\cosh(\ell_{13}-\ell_{12}))}\times$$
$$\sqrt{(\cosh\ell_{24}-\cosh(\ell_{14}+\ell_{12}))(\cosh\ell_{24}-\cosh(\ell_{14}-\ell_{12}))}$$

If we put every edge length to be equal $\ell_{ij} = a$ then we get a particular case of a regular hyperbolic tetrahedron.

Corollary 2. *Let $T = T(a)$ be a regular hyperbolic tetrahedron and all of its edge lengths are equal to $a, a \geq 0$. Then the volume $V = V(T)$ is given by the formula*

$$V = \int\limits_0^a \frac{3\,t\,\sinh t\,dt}{(1+2\cosh t)\sqrt{(\cosh t+1)(3\cosh t+1)}}.$$

A regular case was done before in several works (see, e.g., formula (2.5) in [3]). Corollary 2 completely coincides with them.

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References

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