Recent results on pronormality of subgroups of odd index in finite groups

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We consider only finite groups. A subgroup H of a group G is said to be *pronormal* in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for each $g \in G$. Some of well-known examples of pronormal subgroups are the following: normal subgroups; maximal subgroups; Sylow subgroups; Sylow subgroups of proper normal subgroups; Hall subgroups of solvable groups. Some problems in finite group theory, combinatorics, and permutation group theory were solved in terms of the pronormality, see, for example [1, 11, 12].

In 2012, E. Vdovin and D. Revin [13] proved that the Hall subgroups (when they exist) are pronormal in all simple groups and, guided by the analysis in their proof, they conjectured that any subgroup of odd index of a simple group is pronormal in this group. This conjecture was disproved in [5, 6]. In [4–7], finite simple groups in which all the subgroups of odd index are pronormal were studied. More detailed surveys of investigations on pronormality of subgroups of odd index in finite (not necessary simple) groups could be found in survey papers [3, 8]. These surveys contain new results and some conjectures and open problems. One such open problem is to complete classification of finite simple groups in which all the subgroups of odd index are pronormal. One more open problem involves the classification of direct products of nonabelian simple groups in which the subgroups of odd index are pronormal. A detailed motivation for these problems was provided in [2]. Note that there are examples of nonabelian simple groups G such that all the subgroups of odd index are pronormal in G, but the group $G \times G$ contains a non-pronormal subgroups of odd index (see [2, Proposition 1]).

In this talk, we discuss a recent progress in investigations on pronormality of subgroups of odd index in finite groups. In particular, we have obtained the complete classification finite simple exceptional groups of Lie type in which the subgroups of odd index are pronormal [9] and have proved that the subgroups of odd index are pronormal in a direct product G of simple symplectic groups over fields of odd characteristics if and only if the subgroups of odd index are pronormal in each direct factor of G [10]; moreover, deciding the pronormality of a given subgroup of odd index in the direct product of simple symplectic groups over fields of odd characteristics is reducible to deciding the pronormality of some subgroup H of odd index in a subgroup of a group $\prod_{i=1}^{t} Z_i \wr Sym_{n_i}$, where $Z_i \in \{1, C_3\}$ for each i, such that H projects onto $\prod_{i=1}^{t} Sym_{n_i}$, and we have obtained a criterium of pronormality of such a subgroup in such a group [10].[1] These investigations give a rise to researches on effective algorithms (to be implemented in GAP) for deciding the pronormality of a subgroup of odd index in a finite simple group. [2]

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