Combinatorial limits and their applications in extremal combinatorics

Part 1

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Graph limits

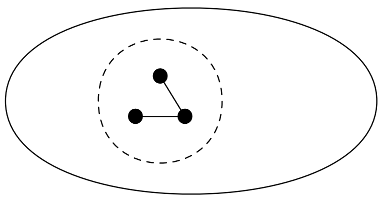
- large networks $\approx$ large graphs
  how to represent? how to model? how to generate?
- concise (analytic) representation of large graphs
  we implicitly use limits in our considerations anyway
- mathematics motivation – extremal graph theory
  What is a typical structure of an extremal graph?
  calculations avoiding smaller order terms
- this course: dense graphs ($|E| = \Omega(|V|^2)$)
- convergence vs. analytic representation

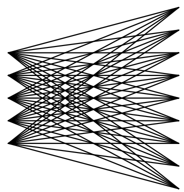
Overview of the course

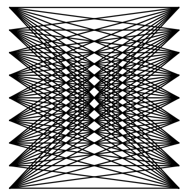
- Introduction – dense graph convergence, graph limits
  limits of dense graphs, permutations and hypergraphs
- Graph and permutation quasirandomness via limits
- Flag algebra method and its relation to graph limits
  applications of flag algebra method in combinatorics
- Computer assisted use of flag algebras via SDP
  various examples of the use of the method

Dense graph convergence

- convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
- extendable to other discrete structures





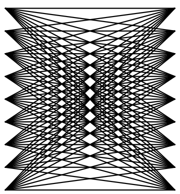


Convergent graph sequences

- complete graphs K_n
- complete bipartite graphs $K_{\alpha n,n}$
- Erdős-Rényi random graphs $G_{n,p}$
- any sequence of graphs with bounded maximum degree
- any sequence of planar graphs

Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$
  measurable symmetric function, i.e. $W(x,y) = W(y,x)$
- “limit of adjacency matrices” (very imprecise)
- points of $[0,1] \approx$ vertices, values of $W \approx$ edge density



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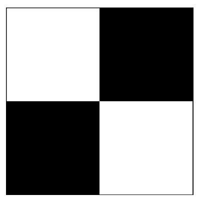

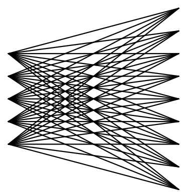

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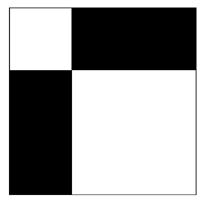
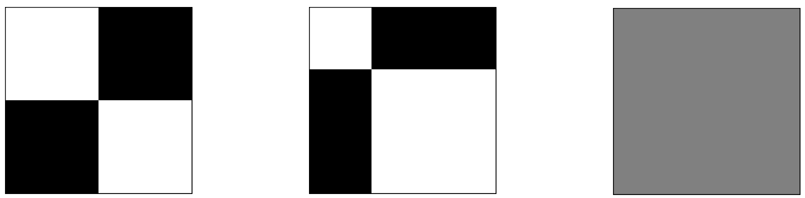


W-RANDOM GRAPHS

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  sample n random points $x_i \in [0,1] \approx$ vertices
  join two vertices by an edge with probability $W(x_i, x_j)$
- density of a graph H in a graphon W
  $d(H,W) =$ prob. $|H|$-vertex W-random graph is H



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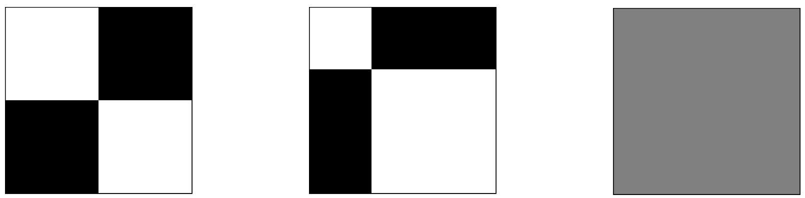
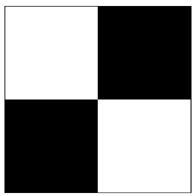
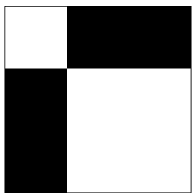
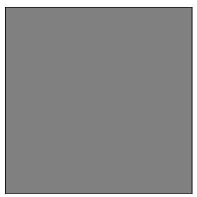
$$\frac{|H|!}{|\mathrm{Aut}(H)|} \int_{[0,1]^{|H|}} \prod_{v_i v_j} W(x_i, x_j) \prod_{\overline{v_i v_j}} (1 - W(x_i, x_j)) \, \mathrm{d}x_1 \cdots x_n$$



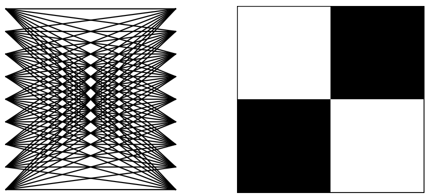
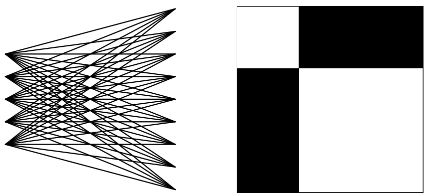
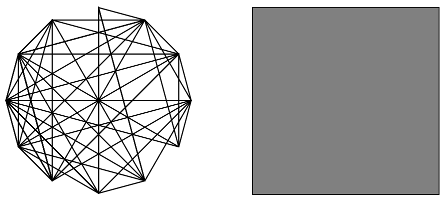
W-RANDOM GRAPHS

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- $d(H,W)$ = expected density of H in a W-random graph
- $d(K_2,W) = \frac{1}{3}d(\overline{K_{1,2}},W) + \frac{2}{3}d(K_{1,2},W) + d(K_3,W)$
  Why? Integral. Random experiment.

W-random graphs

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- $d(H,W)$ = expected density of H in a W-random graph
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$

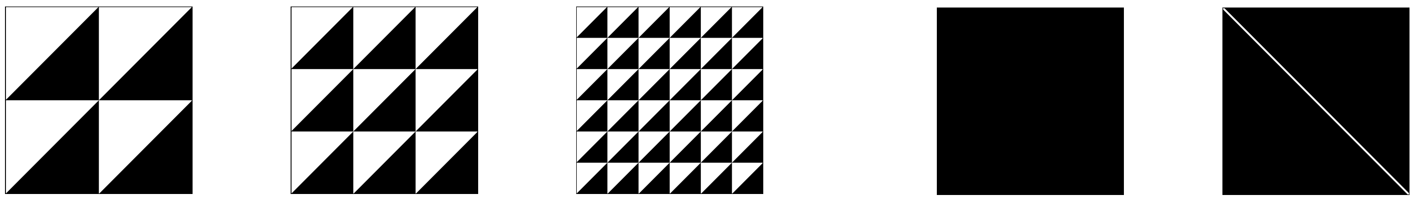
Questions?

Graphons as limits

- Does every convergent sequence have a limit?
- Uniqueness of a graphon representing a sequence.
- Is every graphon a limit of convergent sequence?

Uniqueness of the limit

- $W^{\varphi}(x,y) := W(\varphi(x),\varphi(y))$ for $\varphi : [0,1] \to [0,1]$
- $d(H,W) = d(H,W^{\varphi})$ if φ is measure preserving
- Theorem (Borgs, Chayes, Lovász)
  If $d(H,W_1) = d(H,W_2)$ for all graphs H,
  then there exist measure preserving maps φ_1 and φ_2
  such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere.



Questions?

Thank you for your attention!