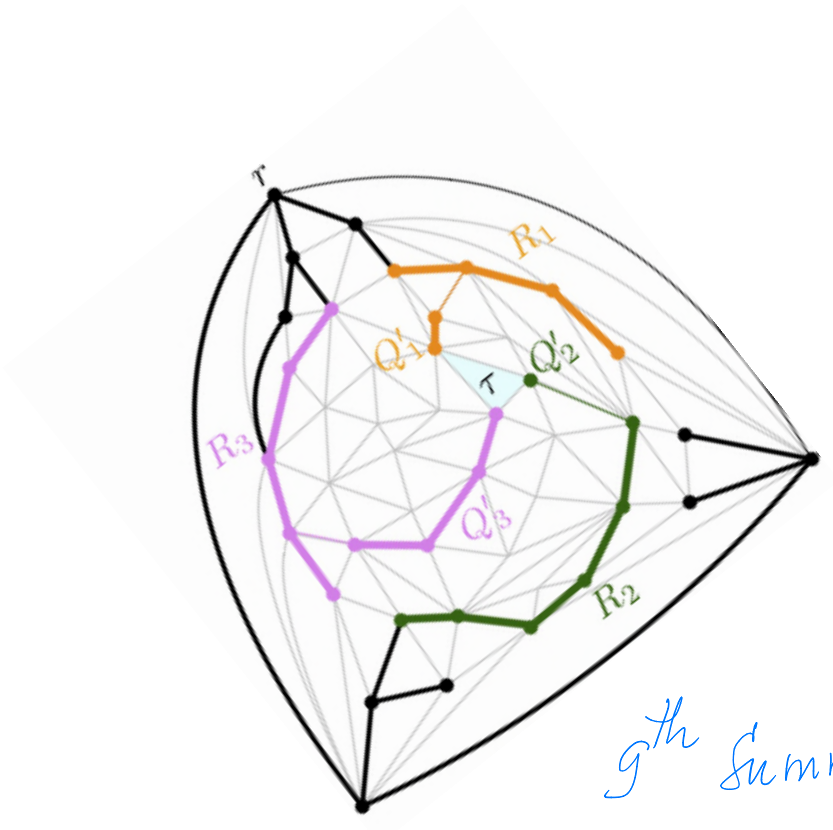
Layered partitions with applications



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APPLICATIONS

- queue layouts [1992]
  & 3D grid drawings [2002]
- non-repetitive graph colourings [2002]
- p-centered colourings
- clustered colourings

  ⋮

- simple proofs of known results, graph products, . . .

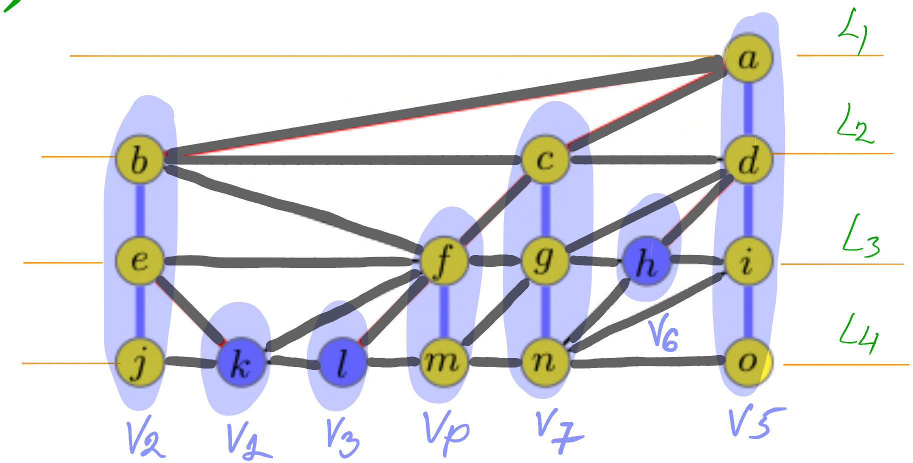
simple proofs + better bounds

tool: LAYERED PARTITIONS

layering $\mathcal{L} = \langle L_1, L_2, \dots \rangle$ of G

partition $\mathcal{P} = \{V_1, \dots, V_p\}$ of $V(G)$

layered partition: a pair $(\mathcal{L}, \mathcal{P})$



tool: LAYERED PARTITIONS

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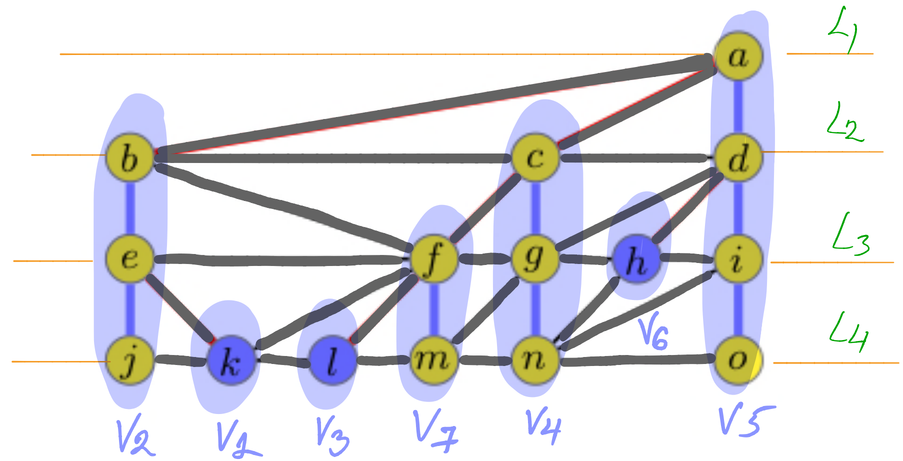
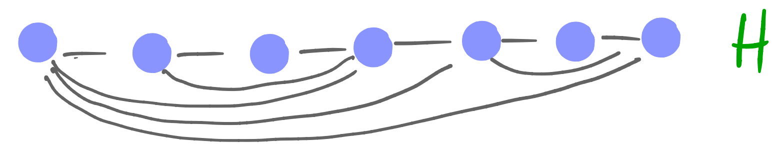




layered width : $\max\{|L \cap P| : L \in \mathcal{L}, P \in \mathcal{P}\}$ & quotient graph $H = G/\mathcal{P}$

WANT

layered H-partitions of G with :

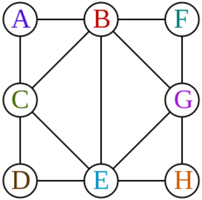
small (constant) layered width

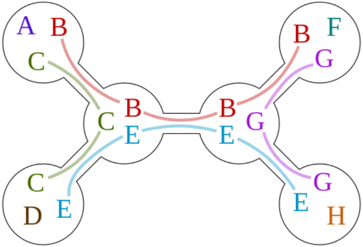
+

nice quotient graph H

bounded tree width

a tree decomposition represents each vertex as a subtree of a tree T so that the subtrees of adjacent vertices intersect in T





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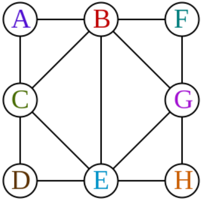
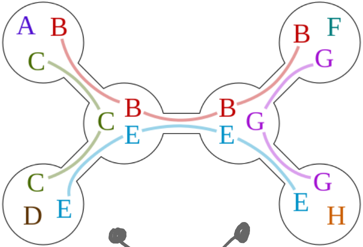
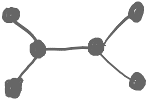
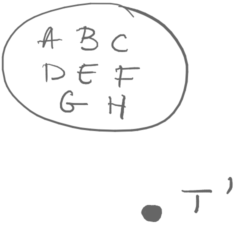


tree-width := maximum bag size -1

↳ measure of how 'tree-like' a graph is







which graph classes have it?

- apex minor-free graphs (planar, bounded genus, ...)

  } minor closed

  [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

- k-planar, d-map graphs ...

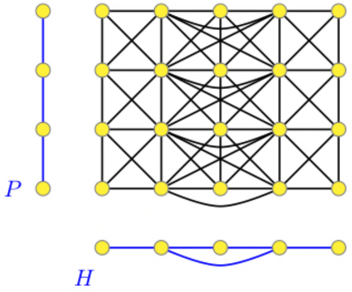
  } non-minor closed

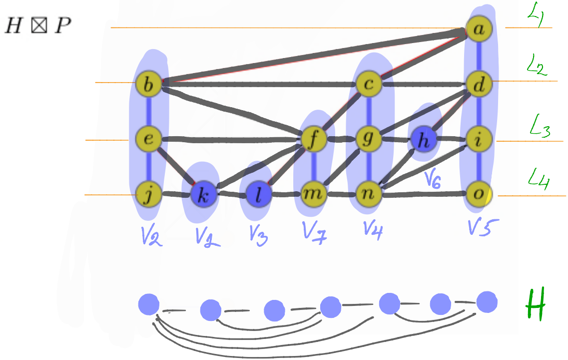
  [D., Morin, Wood] 2019

structure of planar graphs

Corollary

every planar graph G is the subgraph of $H \boxtimes P$
for some graph H with treewidth $\leqslant 8$ and some path P





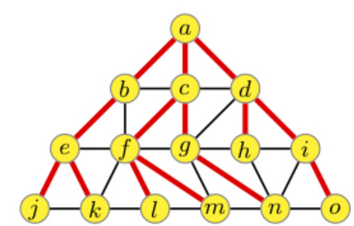
First there was...

Th: [Mi. Pilipczuk, Siebertz] 2018
Every planar graph G has a partition $\mathcal{P}$ into geodesics s.t. $G/\mathcal{P}$ has treewidth ≤ 8.

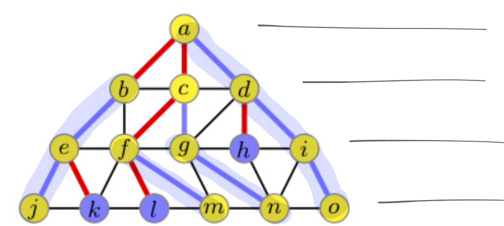
Main theorem

Th: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

For every BFS spanning tree T of a planar graph G, $\exists$ a partition $\mathcal{P}$ into vertical paths s.t. treewidth$(G/\mathcal{P}) \leq 8$.



– gives layered H-partitions of layered width 1 & $tw(H) \leq 8$



Main theorem: Proof

Th: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

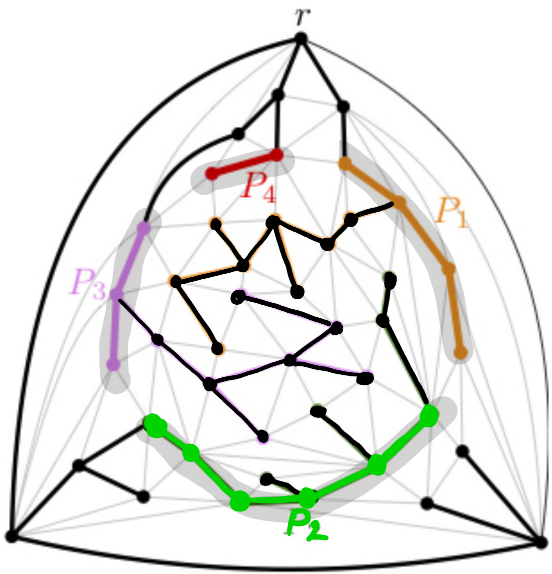
For every BFS spanning tree T of a planar graph G, $\exists$ a partition $\mathcal{P}$ into vertical paths s.t. treewidth$(G/\mathcal{P}) \leq 8$.

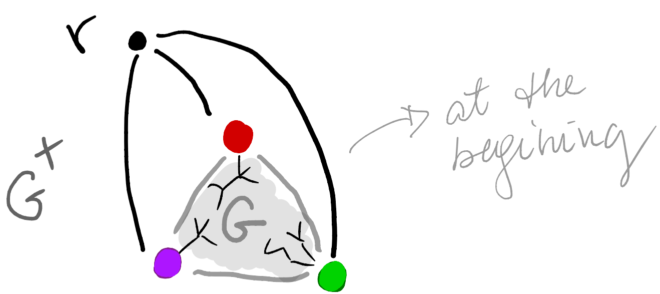
Proof: Partitioning Planar Graphs

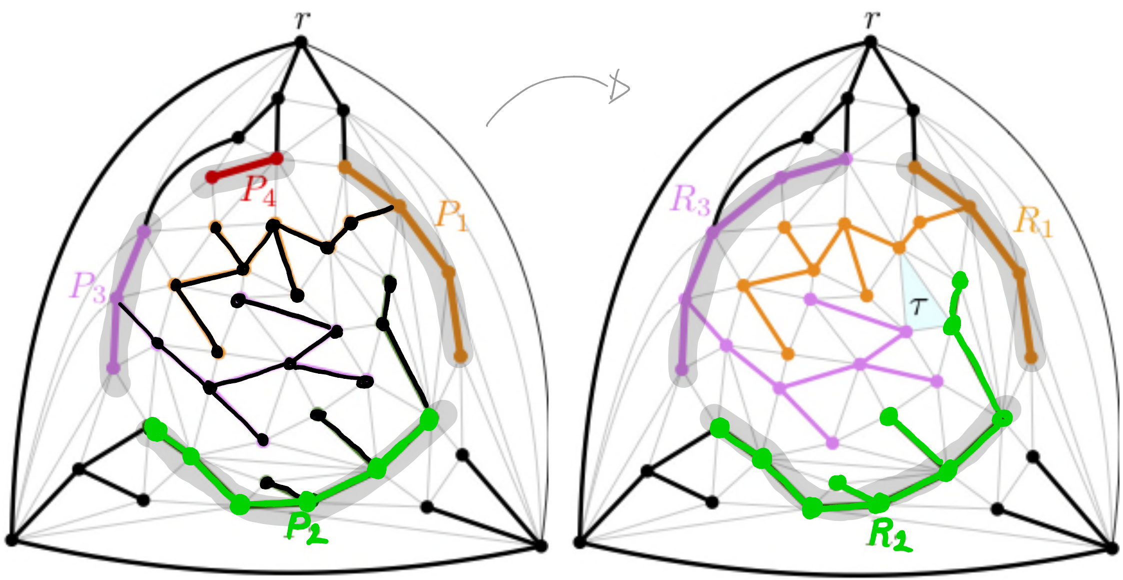
Key lemma. Suppose

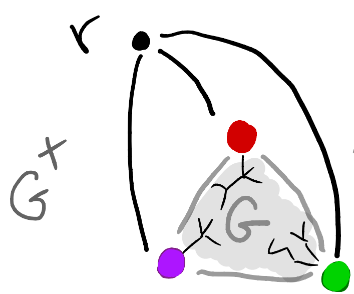
- G^+ plane triangulation
- T rooted spanning tree of G^+ with root on outer-face
- cycle C partitioned into vertical paths $P_1, \dots, P_k$, with $k \leq 6$
- G near-triangulation consisting of C and everything inside.

Then G has a partition $\mathcal{P}$ into vertical paths where $P_1, \dots, P_k \in \mathcal{P}$ s.t. $= G/\mathcal{P}$ has a tree-decomposition in which every bag has size at most 9 and some bag contains all vertices corresponding to $P_1, \dots, P_k$.


r
P_4
P_1
P_3
P_2


r
G^+
G
at the begining





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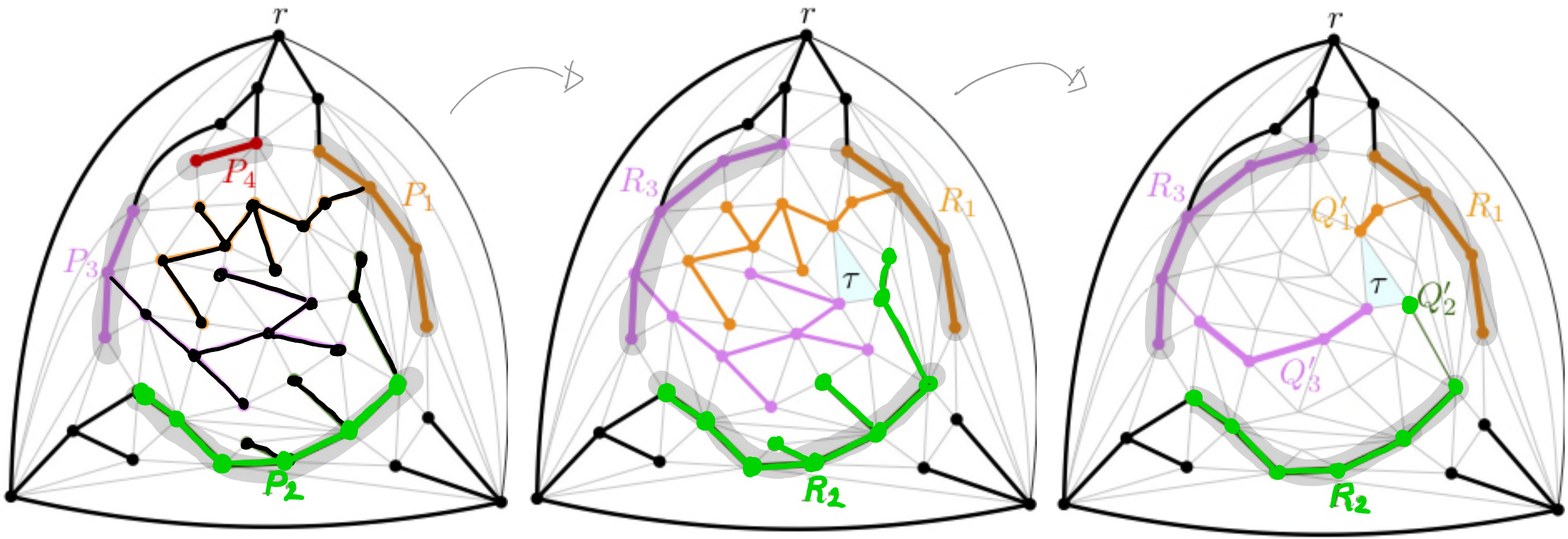
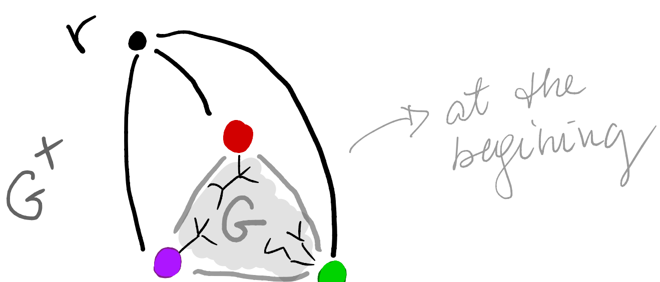
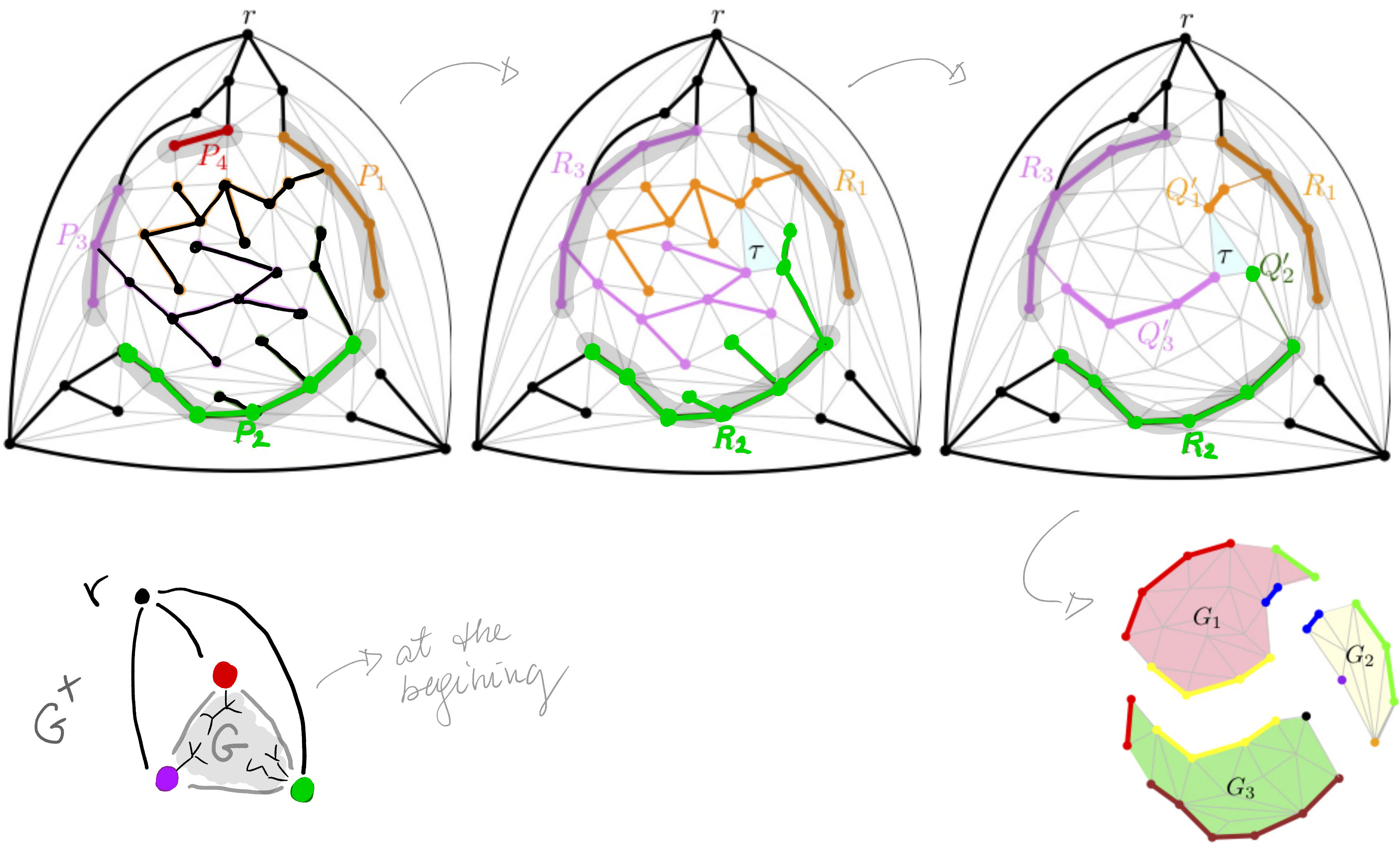


r
P4
P1
P3
P2
r
R3
R1
τ
R2
r
R3
Q'1
R1
τ
Q'2
Q'3
R2



r
G+
G
at the begining



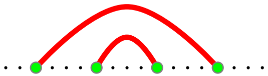
r
P_4
P_1
P_3
P_2
r
R_3
R_1
τ
R_2
r
R_3
Q'_1
R_1
τ
Q'_2
Q'_3
R_2
r
G^+
G
at the begining
G_1
G_2
G_3

Applications

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19 Planar graphs have bounded queue-number

Dujmović, Esperet, J., Walczak, Wood '19 Planar graphs have bounded nonrepetitive chromatic number *k*-queue layout of graph G:

- vertex ordering $v_1, \ldots, v_n$ of G
- partition $E_1, \ldots, E_k$ of $E(G)$ such that no two edges in E_i are nested



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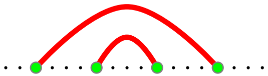

queue-number $\text{qn}(G) :=$ minimum k such that G has a k-queue layout

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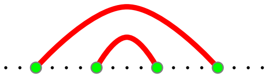

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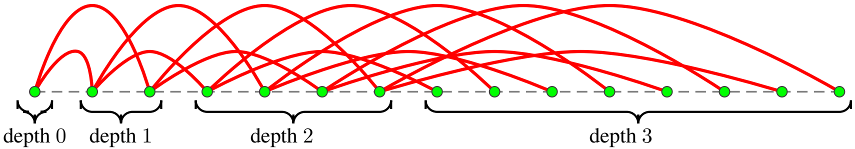
example qn(tree)=1



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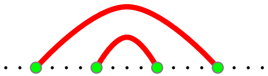

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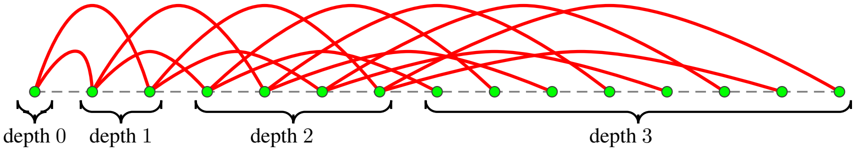


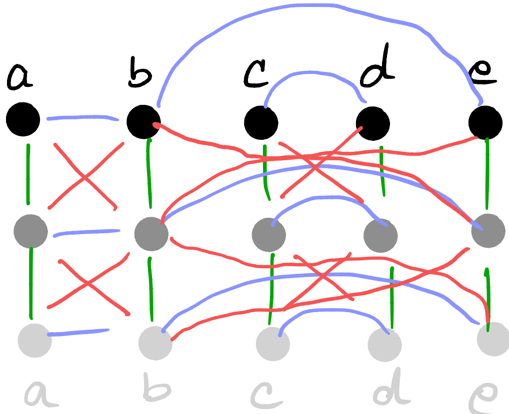
open problem [Heath, Leighton, Rosenberg '92]
do planar graphs have bounded queue-number?

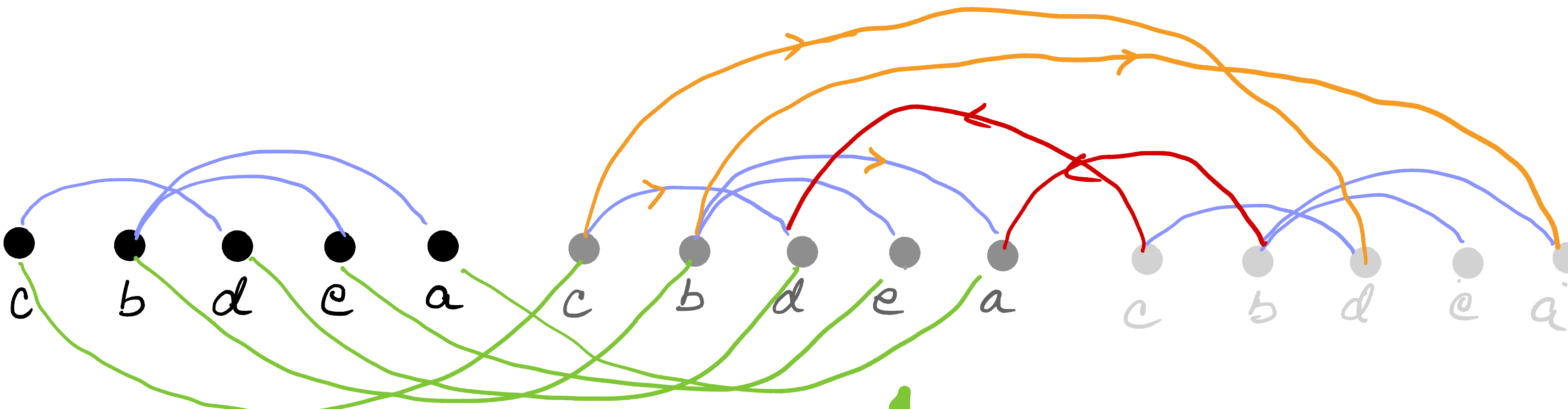
Using layered H-partitions of layered width ℓ

Lemma: [DJMMUW '19 & Wood '08]

$$gn(G) \leq 3 \cdot \ell \cdot gn(H) + \lfloor \tfrac{3}{2} \ell \rfloor$$

Proof by picture





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Using layered H-partitions of layered width ℓ

lemma: [DJMMUW '19 & Wood '08]

$$qn(G) \leq 3 \cdot \ell \cdot qn(H) + \lfloor \tfrac{3}{2}\ell \rfloor$$

KNOWN:

- bounded treewidth graphs have bounded queue-number [D., Morin, Wood '08]
- $qn(H) \leq 2^{tw(H)} - 1$ [Wiechart '18]

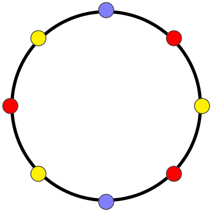
theorem:

$$qn(\text{planar } G) \leq 3(2^8 - 1) + 1 = 766$$

[D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

nonrepetitive colourings

a colouring of a path $v_1, \ldots, v_{2t}$ is repetitive
if $\text{col}(v_i) = \text{col}(v_{t+i})$ for $i \in \{1, \ldots, t\}$



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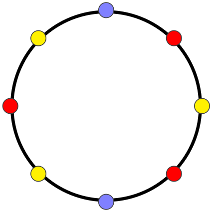

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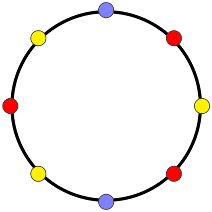

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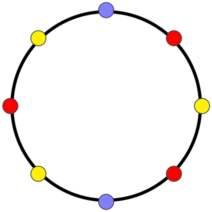

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1906

- $\pi(\text{path}) \leqslant 3$ [Thue ~~'06~~]

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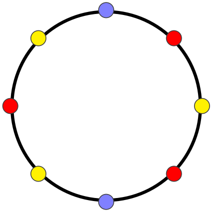

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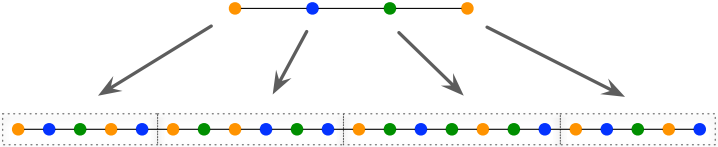

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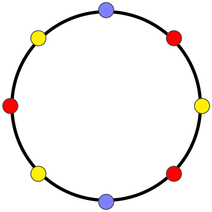

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- $\pi(\text{path}) \leqslant 3$ [Thue '~~06~~ 1906]
- $\pi(\text{max degree } \Delta \text{ graph}) \leqslant O(\Delta^2)$
[Alon, Grytczuk, Hałuszczak, Riordan '02]

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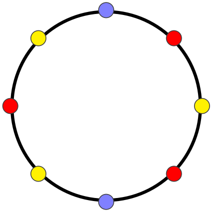

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open problem [Alon, Grytczuk, Hałuszczak, Riordan '02]
do planar graphs have bounded π?

Using layered H-partitions of layered width ℓ

- $\pi(H) \leqslant 4^{\mathrm{tw}(H)}$ [Kündgen & Pelsmajer '08]

Using layered H-partitions of layered width ℓ

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a colouring is strongly nonrepetitive if for every repetitively coloured lazy walk $v_1, \dots, v_{2t}$, there exists $i \in \{1, \dots, t\}$ such that $v_i = v_{i+t}$

$\pi^*(G)$:= min. # colours in a strongly nonrepetitive colouring of G

Using layered H-partitions of layered width ℓ

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$\pi^*(G)$:= min. # colours in a strongly nonrepetitive colouring of G

lemma: [D., Esperet, Joret, Walczak, Wood] 2019

$$\pi^*(G) \leq 4 \cdot \ell \cdot \pi^*(H)$$

Cor:

$$\pi(\text{planar } G) \leq \pi^*(G) \leq 4 \cdot \ell \cdot \pi^*(H) \leq 4 \cdot \ell \cdot 4^{tw(H)} \leq 4 \cdot 4^8 = 262{,}144$$

Improved bounds and generalizations

Variant of the main theorem

Th: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

Every planar graph G has a layered partition $\mathcal{P}$ of layered width 3 where $tw(\underbrace{G/\mathcal{P}}_{H}) \leq \underline{3}$.

parts not connected

planar

best possible

APPLICATIONS

Cor: [D., Esperet, Joret, Walczak, Wood] 2019

$$\pi(\text{planar } G) \leq 4 \cdot \ell \cdot 4^{tw(H)} \leq 4 \cdot 3 \cdot 4^3 = 768$$

Cor: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

$$qn(\text{planar } G) \leq 3 \cdot \ell \cdot (2^{tw(H)} - 1) + \lfloor \tfrac{3\ell}{2} \rfloor \leq 3 \cdot 3 \cdot (2^3 - 1) + 4 \leq 71$$

APPLICATIONS

Cor: [D., Esperet, Joret, Walczak, Wood] 2019

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$$\le 3 \cdot \ell \cdot gn(H) + \lfloor \tfrac{3}{2} \ell \rfloor = 3 \cdot 3 \cdot \underline{5} + 4 = 49$$

APPLICATIONS

Cor: [D., Esperet, Joret, Walczak, Wood] 2019

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$$\le 3\cdot \ell \cdot gn(H) + \left\lfloor \tfrac{3}{2}\ell \right\rfloor = 3\cdot 3\cdot \underline{\underline{5}} + 4 = 49$$

theorem: [Felsner, Micek, Schroeder] 2019+

Planar graphs have p-centered colourings with $O(p^3 \lg p)$ col

Generalization of the main theorem

[D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

Th: Every graph G of Euler genus g has a layered partition $\mathcal{P}$ of layered width $\max\{2g, 3\}$ where $tw(\underbrace{G/\mathcal{P}}_{H}) \leq 4$.

apex minor free

Th: $\forall K\ \exists k, a$ s.t every K-minor free graph G can be obtained by clique-sums of graphs $G_1, \dots, G_t$ s.t for $i \in \{1, \dots, t\}$

$$G_i \subseteq (H_i \boxtimes P_i) + K_a$$

for some H_i with $tw(H_i) \leq k$ and some path P_i

APPLICATIONS

th: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019
graphs excluding a fixed minor have bounded queue #.

th: [D., Esperet, Joret, Walczak, Wood] 2019
graphs excluding a fixed topological minor have bounded non-rep χ

- low treewidth colourings

simple proofs with better bounds

OPEN PROBLEMS

- H something other than bounded treewidth
- algorithmic and other applications
- proof without using structure theorem

- queue-number, non-repetitive χ of graphs that have strongly sub-linear separators

structure?

the end