Combinatorial limits and their applications in extremal combinatorics

Part 3

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Rogla July 2019

Permutations

- permutation of order n: order on numbers $1, \ldots, n$

  subpermutation: $4\underline{53}21\underline{6} \longrightarrow 213$

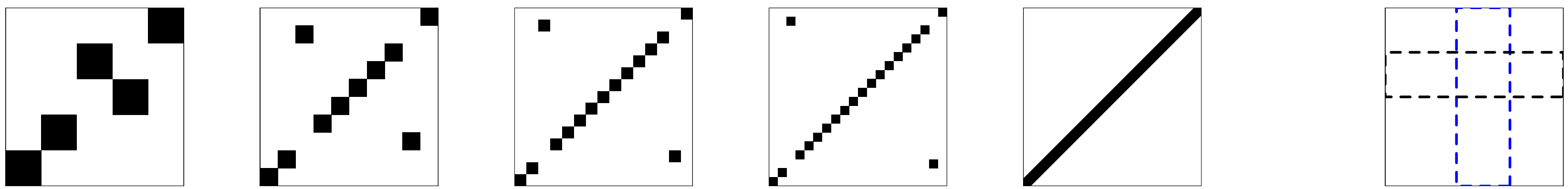
- density of a permutation π in a permutation Π:

$$d(\pi, \Pi) = \frac{\#\text{ subpermutations of } \Pi \text{ that are } \pi}{\#\text{ all subpermutations of order } \pi}$$

- $(\Pi_j)_{j\in\mathbb{N}}$ convergent if $\exists \lim_{j\to\infty} d(\pi, \Pi_j)$ for every π

Representation of a limit

- probability measure μ on $[0,1]^2$ with unit marginals
  $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a$
  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio
- μ-random permutation
  choose n random points, x- and y-coordinates



Quasirandom graphs

- Thomason, and Chung, Graham and Wilson (1980's)
- a sequence G_i is quasirandom if $d(H, G_i) \approx d(H, G_{n,p})$
  G_i converges to the constant graphon W_p
- $d(H, G_i) \to d(H, W_p)$ for every H if and only if
  $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$
- $h(H, G) =$ prob. inj. map $H \to G$ is a homomorphism
  $h(\cdot, G)$ and $d(\cdot, G)$ for determine each other

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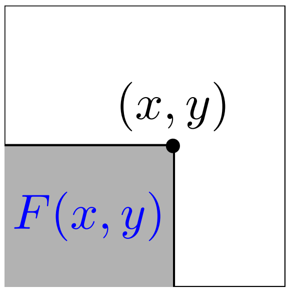
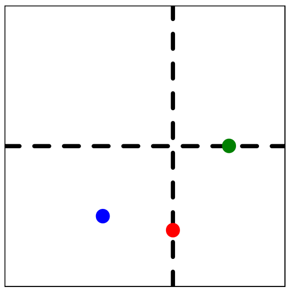
  $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$
- $W \equiv p \Leftrightarrow h(K_2, W) = p$ and $h(C_4, W) = p^4$

QUASIRANDOM PERMUTATIONS

- property $P(k)$ of $(\Pi_j)_{j\in\mathbb{N}}$: $d(\sigma,\Pi_j) \to 1/k!$ for $\forall\, \sigma \in S_k$
- Question (Graham): Is there k_0 such $\forall k\; P(k_0) \Rightarrow P(k)$?
- Theorem (K., Pikhurko): yes, $k_0 = 4$; best possible

$$\frac{1}{81} = \left(\int F(x,y)xy\ \mathrm{d}x\mathrm{d}y\right)^2 \le \frac{1}{9}\int F(x,y)^2\mathrm{d}x\mathrm{d}y = \frac{1}{81}$$



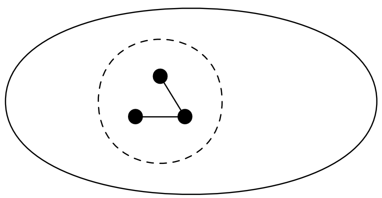
$\int F(x,y) =$  $=$

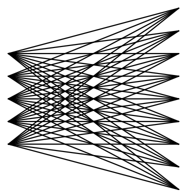
123 123 132
132 213 213
231 312 321

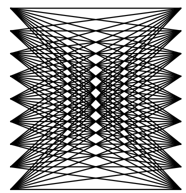
Questions?

Dense graph convergence

- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is convergent
  if $d(H, G_n)$ converges for every H
- examples of convergent sequences:
  complete and complete bipartite graphs K_n and $K_{\alpha n, n}$
  Erdős-Rényi random graphs $G_{n,p}$

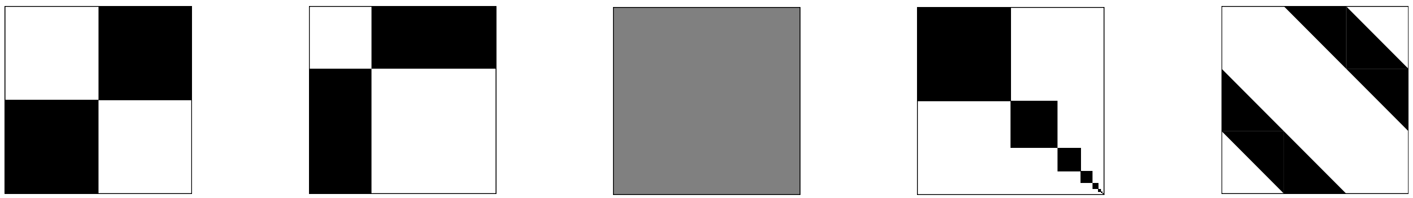






Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$



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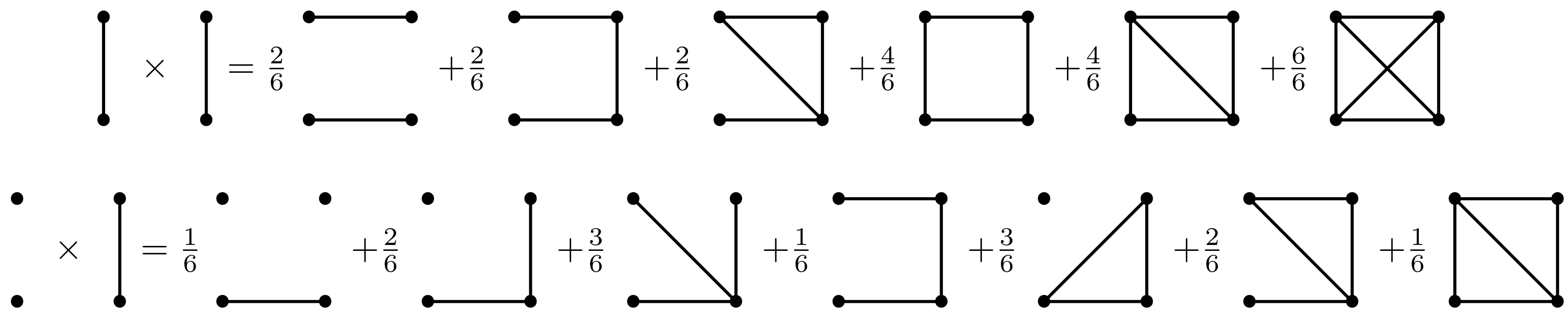
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$
- W-random graphs converge to W with probability one
- every convergent sequence of graphs has a limit

Flag algebras

- the flag algebra method independent of graph limits
  we introduce the method using graphons for simplicity
- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar
- homomorphism f_W from $\mathcal{A}$ to $\mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
- examples: $f_W(K_2) = d(K_2, W)$
  $f_W(K_2 - K_3) = d(K_2, W) - d(K_3, W)$

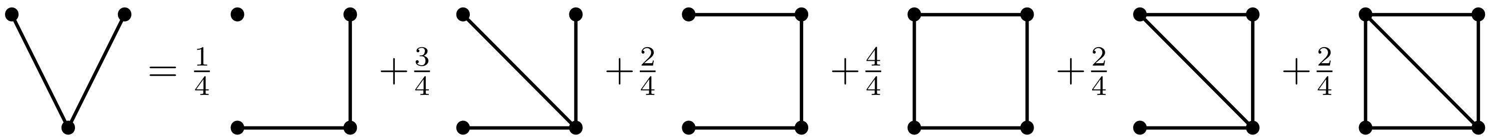
Multiplication

- defined $f_W(H) := d(H, W)$ and extended linearly
- aim: define multiplication on $\mathcal{A}$ preserved by f_W
  $f_W(H_1 \times H_2) = f_W(H_1) \cdot f_W(H_2)$
- $H_1 \times H_2 = \sum_H \frac{|\{(A,B) | V(H) = A \cup B, H[A] \cong H_1, H[B] \cong H_2\}|}{\binom{|H_1|+|H_2|}{|H_1|}} H$



Kernel of the map

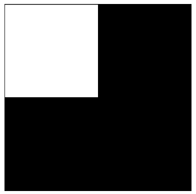
- defined $f_W(H) := d(H, W)$ and extended linearly
- $\mathrm{Ker}(f_W)$ always contains certain elements
  $f_W(K_2) = \frac{1}{3} f_W(\overline{K_{1,2}}) + \frac{2}{3} f_W(K_{1,2}) + \frac{3}{3} f_W(K_3)$



- let $\mathcal{A}'$ be the space generated by $H - \sum_{H'} d(H, H')H'$
  $\mathcal{A}' \subseteq \mathrm{Ker}(f_W) \Rightarrow$ homomorphism $f_W : \mathcal{A}/\mathcal{A}' \to \mathbb{R}$

Rooted homomorphisms

- consider a graph G with a distinguish vertex (root)
  a random sample always includes the root
- algebra $\mathcal{A}^\bullet$ on combinations of rooted graphs
- rooted graphon $\to$ a homomorphism from $\mathcal{A}^\bullet$ to $\mathbb{R}$
  random choice of the root x_0 $\to$ probability distribution on homomorphisms f^{x_0} from $\mathcal{A}^\bullet$ to $\mathbb{R}$

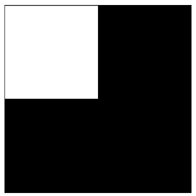


$$f^\bullet(K_2^\bullet)=1/2,\ f^\bullet(\overline{K_2^\bullet})=1/2,\ f^\bullet(K_3^\bullet)=1/4,\ \ldots$$

$$f^\bullet(K_2^\bullet)=1,\ f^\bullet(\overline{K_2^\bullet})=0,\ f^\bullet(K_3^\bullet)=3/4,\ \ldots$$

Rooted homomorphisms

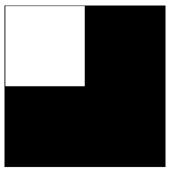
- algebra $\mathcal{A}^\bullet$ of combinations of rooted graphs
  random choice of the root x_0 $\to$ probability distribution on homomorphisms f^{x_0} from $\mathcal{A}^\bullet$ to $\mathbb{R}$
- the value $f_W^{x_0}(H)$ for H with root v_0 is $\frac{k!}{|\mathrm{Aut}^\bullet(H)|}\times$
  $$\int \prod_{v_iv_j \in E(H)} W(x_i,x_j) \prod_{v_iv_j \notin E(H)} (1-W(x_i,x_j))\, \mathrm{d}x_1 \cdots x_k$$



$$\frac{f^\bullet(K_2^\bullet)=1/2,\ f^\bullet(\overline{K_2^\bullet})=1/2,\ f^\bullet(K_3^\bullet)=1/4,\ \ldots}{f^\bullet(K_2^\bullet)=1,\ f^\bullet(\overline{K_2^\bullet})=0,\ f^\bullet(K_3^\bullet)=3/4,\ \ldots}$$

General rooted graphs

- fix a graph R with vertices $r_1, \ldots, r_k$
  algebra $\mathcal{A}^R$ of combinations of R-rooted graphs
- random homomorphism f^R from $\mathcal{A}^R$ to $\mathbb{R}$
  random choice of the roots $x_1, \ldots, x_k$
  the roots do not induce $R \Rightarrow f^R \equiv 0$
  otherwise, sampling $|H| - k$ vertices $\Rightarrow$ prob. $f^R(H)$



$f^{K_2}(K_3^{K_2}) = 0,\ f^{K_2}(K_4^{K_2}) = 0,\ f^{K_2}(K_{1,2}^{K_2}) = 0,\ \ldots$

$f^{K_2}(K_3^{K_2}) = 1/2,\ f^{K_2}(K_4^{K_2}) = 1/4,\ f^{K_2}(K_{1,2}^{K_2}) = 1/2,\ \ldots$

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Questions?

Thank you for your attention!