Combinatorial Methods in Group Theory (and Group-theoretic Methods in Combinatorics)

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Outline of topics

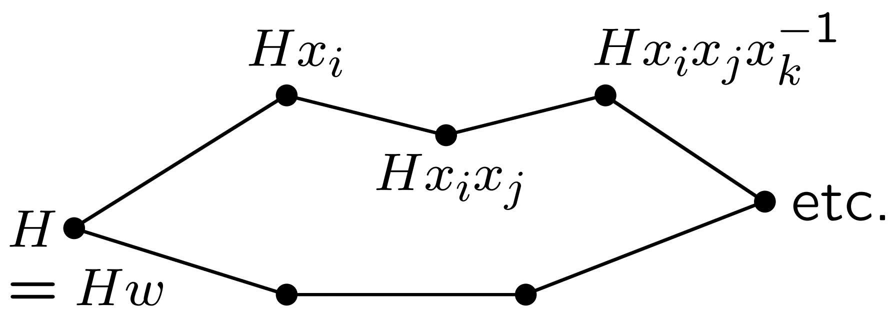
1. Basic applications of counting
2. Methods for generating random elements of a group
3. Cayley graphs
4. Schreier coset graphs and their applications
5. Back-track search to find small index subgroups
6. Double-coset graphs and some applications
7. Möbius inversion on lattices and applications

Copies of slides can be made available by email or USB stick.

Further properties of the Schreier coset graph $\Sigma(G, X, H)$

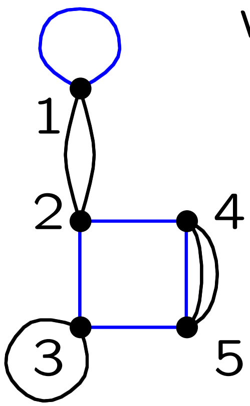
First, recall that every circuit in Σ based at the vertex H gives an element of the subgroup H, written as a word on X.

Why? Any walk in Σ corresponds to a word $w = w(X)$ in the generators of G, and such a path from H is closed whenever $Hw = H$, which occurs if and only if $w \in H$.



It follows that a set of generators for H can be created from the circuits in Σ based at the vertex labelled H.

Why? On one hand, all of the words coming from those circuits are elements of H. On the other hand, specifying all of them is enough to define the graph Σ, and hence the action on G on cosets, and hence H (as point-stabiliser).

e.g.  with $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

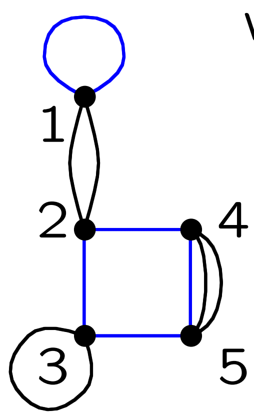
gives $H = \langle\, y, x^2, xyxy^{-1}x^{-1}, xy^4x^{-1}, xy^{-1}xy^{-2}x, xy^2xyx \,\rangle$.

Q: How/when do we know that we have enough generators?

Reidemeister-Schreier Theory (when $|G:H|$ is finite)

First, a Schreier transversal for H in G is defined as a complete set $T = \{w_1, w_2, \ldots, w_n\}$ of representatives of the $n = |G:H|$ right cosets Hg of H, each expressed as a word in the generating set X, with the Schreier property that for each $w \in T$, every 'initial sub-word' of w also lies in T.

e.g.



with $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

gives $T = \{1, x, xy, xy^{-1}, xy^2\}$ as a Schreier transversal.

Generally, a Schreier transversal for H in G corresponds to a spanning tree for the coset graph Σ.

Why?

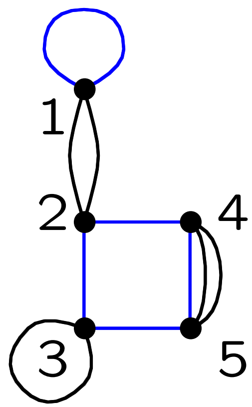
In any spanning tree T for Σ, we can label each vertex v with the product of the elements of X that label the edges of a path from 1 to v in T. The resulting n words form a transversal for H in G with the Schreier property, since any 'initial sub-word' of any one of them traces an initial sub-path from 1 to some vertex u of the corresponding path.

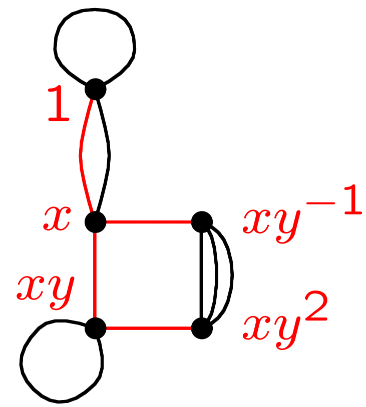
Exercise: Convince yourself that the converse is also true.

Example (as before)

Perm. representation $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

Schreier transversal $T = \{1, x, xy, xy^{-1}, xy^2\}$





Check: Every 'initial sub-word' of each $w \in T$ also lies in T

Reidemeister-Schreier Theory (cont.)

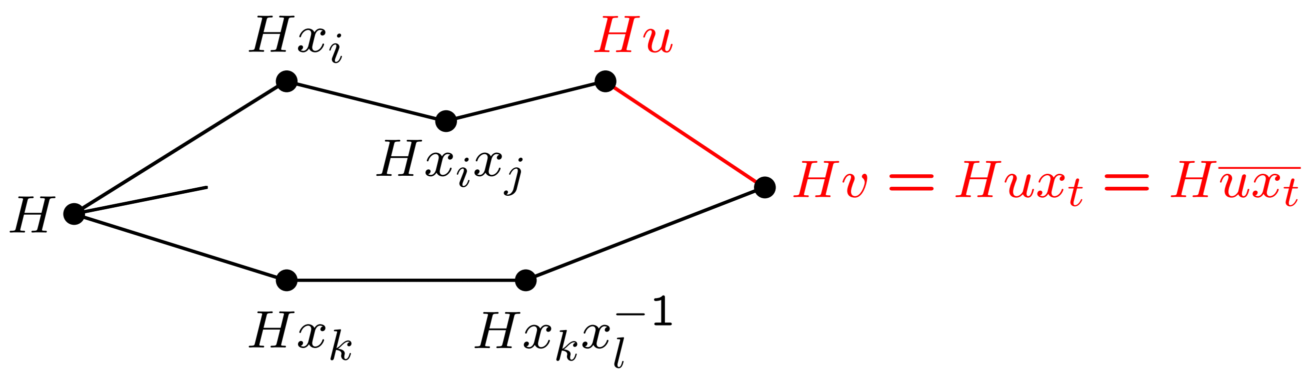
Next, suppose T is a Schreier transversal for H in G, and for every element $g \in G$, denote by $\overline{g}$ the representative in T of the (right) coset Hg. In other words, $\overline{g}$ is the unique element of T for which $Hg = H\overline{g}$.

Schreier's Subgroup Lemma: The subgroup H is generated by the set $\{\, ux(\overline{ux})^{-1} : u \in T,\, x \in X \,\}$.

Sketch proof. First, each of the elements $ux(\overline{ux})^{-1}$ lies in H, because $Hux = H\overline{ux}$ for all $u \in T$, $x \in X$.

Indeed the x-edge from vertex Hu to vertex Hux creates a circuit in Σ based at vertex H, running from H to Hu along edges of T, then across to Hux ($= H\overline{ux}$), and back to H along edges of T again.

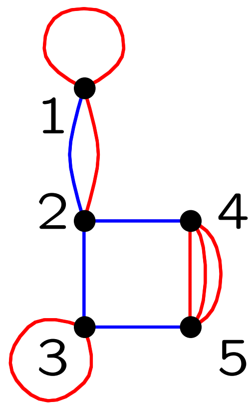
Conversely, every edge of Σ not in T has this form.



Hence these 'Schreier edges' are enough to add back all the edges of Σ, completing Σ, and determining H as the stabiliser of the vertex 'H' in the corresponding permutation representation of G (of degree $n = |G:H|$).

Summary: A Schreier generating-set for H in G corresponds to edges of the coset graph not used in the spanning tree.

Example



with $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

A spanning tree has $5-1=4$ edges; for example, the edges $\{1,2\}$, $\{2,3\}$, $\{2,4\}$ and $\{4,5\}$ giving $\{1, x, xy, xy^{-1}, xy^2\}$ as a Schreier transversal. The total number of edges is 10, so $10-4=$ 6 of them do not lie in the spanning tree.

These give the six non-trivial Schreier generators y, x^2, $xyxy^{-1}x^{-1}$, xy^4x^{-1}, $xy^{-1}xy^{-2}x^{-1}$ and xy^2xyx^{-1} for H.

Corollary: Every subgroup of index n in a k-generator group can be generated by at most $nk - n + 1$ elements.

Proof. The coset graph has up to nk edges, and $n - 1$ of these are used in any spanning tree, so there remain at most $nk - (n - 1)$ edges for the non-trivial Schreier generators. ■

In fact, for a free group (which has no non-trivial relations), this bound is attained: every subgroup of index n in a free group of rank k is free of rank $nk - n + 1$. [See later]

Further application: the Reidemeister-Schreier process

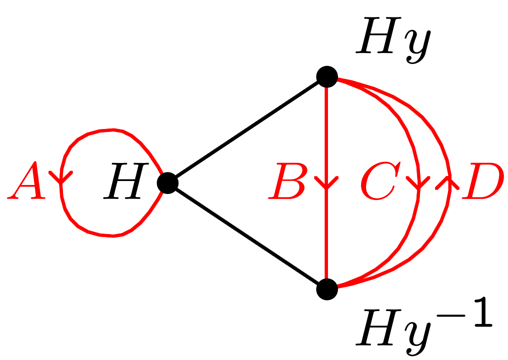
Given a finitely-presented group $G = \langle\, X \,|\, R \,\rangle$, where X is the set of generators and R is the set of defining relations, and a subgroup H of finite index in G, how do we find a presentation for H (in terms of generators and relations)?

This can be done using further Reidemeister-Schreier theory, but the theory is very algebraic and can be difficult to follow. It is easier to explain (and implement) using coset graphs:

1) Construct the coset graph Σ – using the coset table
2) Use a spanning tree to give a Schreier transversal
3) Label the unused edges with Schreier generators
4) Trace each of the relators from R around the coset graph, starting from each of the vertices in turn, to obtain the Reidemeister-Schreier relations for H.

Example 1:

Let $G = \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$, and let H be the stabilizer of 1 in the permutation representation $x \mapsto (2,3)$, $y \mapsto (1,2,3)$:



Schreier generators

$A = x$

$B = y^3$ $(= 1)$

$C = yxy$

$D = y^{-1}xy^{-1}$ $(= C^{-1})$

Relation $x^2 = 1$ gives new relations $A^2 = 1$ and $CD = 1$

Relation $y^3 = 1$ gives new relation $B = 1$

So H has presentation $\langle\, A, C \mid A^2 = 1 \,\rangle$ via $(A, C) = (x, yxy)$.

Example 2:

Let $F_k = \langle\, x_1, x_2, \ldots, x_k \,|\, - \,\rangle$, the free group of rank k, and let H be any subgroup of index n in F_k.

Any spanning tree (with $n-1$ edges) in the Schreier coset graph $\Sigma(F_k, X, H)$ gives a Schreier transversal for H in F_k, and $nk - n + 1$ Schreier generators. And then since F_n is free (which means that it has no non-trivial relations), the Reidemeister-Schreier process gives no relations for H.

Hence H is a free group of rank $nk - n + 1$.

In particular, every subgroup of finite index in a finitely-generated free group is free.

Another application: the Ree-Singerman theorem

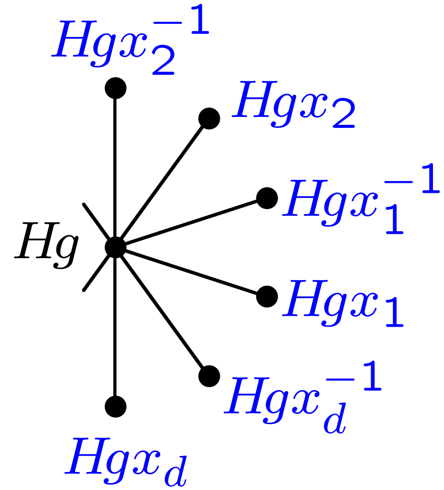
Theorem: Let G be any group generated by permutations $x_1, x_2, \ldots, x_d$ on a set Ω of size n, such that $x_1 x_2 \ldots x_d = 1$ (identity), and let c_i be the number of orbits of $\langle x_i \rangle$ on Ω. If G is transitive on Ω, then $c_1 + c_2 + \cdots + c_d \leq (d-2)n + 2$.

Note that if $d = 2$ then $x_2 = x_1^{-1}$, and the theorem gives $c_1 + c_2 \leq 2$, so $c_1 = c_2 = 1$, making x_1 a single n-cycle.

The case $d = 3$ was proved by Singerman (1970), and the general case ($d \geq 2$) was proved by Ree in 1971, using the theory of Riemann surfaces.

But: it can be proved easily using coset diagrams.

Proof. Embed the coset graph for the given permutation representation in an orientable surface with the edges to neighbours at each vertex Hg ordered naturally as follows:



Here $|V| = n$ and $|E| = nd,$ and we have

(1) a face bounded by x_i-edges for every cycle of each x_i

(2) a face bounded by edges $x_1, x_2, \ldots, x_d$ at each vertex,

and so $|F| = \sum_{1 \leq i \leq d} c_i + n.$

If the permutation representation is transitive, then the coset graph is connected, and then the Euler characteristic χ of the surface into which the graph has been embedded is given by the Euler-Poincaré formula:

$$\chi = |V| - |E| + |F| = n - nd + \sum_{1 \leq i \leq d} c_i + n.$$

But $\chi \leq 2$ for every orientable surface, and so from the above equation(s) it follows that

$$\sum_{1 \leq i \leq d} c_i \leq 2 - 2n + nd = (d-2)n + 2$$

as required. ∎

Application: No transitive group of degree 167 can be generated by elements x, y, z such that $x^2 = y^3 = z^7 = xyz = 1$.

Why not? Assume that there **is** such a group. Let c_x, c_y and c_z be the number of cycles of x, y and z respectively.

Then the smallest possible number of cycles of z occurs when z has $\lfloor \frac{167}{7} \rfloor = 23$ cycles of length 7 and $167 - 23 \cdot 7 = 6$ fixed points, so $c_z \geq 29$. Similarly $c_y \geq 55 + 2 = 57$ and $c_x \geq 83+1 = 84$ (tho we can improve this to $c_x \geq 82+3 = 85$ because $x = (yz)^{-1}$ is an even permutation).

But now $c_x + c_y + c_z \geq 84 + 57 + 29 = 170$ while on the other hand $(d-2)n + 2 = n + 2 = 169$, contradicting the Ree-Singerman theorem. ■

Summary:

Schreier coset graphs

- depict transitive actions of groups
- may help build large quotients of a group from small ones
- can be used to prove certain groups are inifinite
- illustrate/assist the Reidemeister-Schreier process
- help prove the Ree-Singerman theorem.

Next:

How do we find good examples in the first place?

§5. Back-track search for subgroups

Let G be a group generated by $X = \{x_1, x_2, \ldots, x_m\}$.

Recall that right multiplication of right cosets of a subgroup H in G by elements of $X \cup X^{-1}$ can be depicted by a coset table like this:

	x_1	x_2	$\ldots$	x_1^{-1}	x_2^{-1}	$\ldots$
1	2	3		4		
2				1		
3					1	
4	1					
$\vdots$						

Subgroups of index $\leq n$ in G can be found (up to conjugacy) by a systematic enumeration of coset tables with $\leq n$ rows.

The low index subgroups algorithm

- Given $G = \langle X \mid R \rangle$ finitely-presented group
- Algorithm (due to Dietze & Schaps and Sims, in 1970s) finds a representative of each conjugacy class of subgroups of index $\leq n$ (for given n) in G
- Backtrack search through a tree, with nodes at level k corresponding to k-generator (pseudo-)subgroups H
- Enumeration (by Todd-Coxeter) of cosets of H
- Create branches to new nodes at the next level (if necessary) by identifying pairs of cosets: forcing $Hg_i = Hg_j$ is equivalent to adding $g_i g_j^{-1}$ to a set of generators for H

Low index subgroups algorithm (cont.)

- Output includes generators for the subgroup H, and/or permutations induced by generators (in X) on cosets of H

- Schreier's theorem ensures that every subgroup of given index m in $G = \langle X \mid R \rangle$ is finitely-generated (so will be found)

- Conjugates of subgroups found earlier may be eliminated easily (by a test on the coset table)

- This can facilitated by normal ordering of cosets in the coset table – where each 'new' coset is labelled with the smallest unused positive integer

- Example: (PTO)

	x_1	x_2	x_3	x_1^{-1}	x_2^{-1}	x_3^{-1}
1	2	3	4	5		
2				1		
3					1	
4						1
5	1					
⋮						

Cosets 1: H
2: Hx_1
3: Hx_2
4: Hx_3
5: Hx_1^{-1}

Transversal $\{1, x_1, x_2, x_3, x_1^{-1}, \ldots\}$

Processing: e.g. $1x_1^{-1} = 5 \;\Rightarrow 5x_1 = 1$

Forcing coincidences

The key to the standard 'Low index subgroups algorithm' is to define more than n cosets, and then force coincidences between them, using the fact that $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

The algorithm starts with the identity subgroup and attempts to enumerate its right cosets, constructing a partial transversal $\{u_1, u_2, u_3, \dots\}$. Then (or at any stage) if more than n cosets are defined, all possible concidences between two cosets Hu_i and Hu_j are considered, for $1 \le i < j \le n+1$.

Often such a coincidence is found to produce a subgroup H that is conjugate to one found previously, in which case that coincidence is rejected and the next one is looked at.

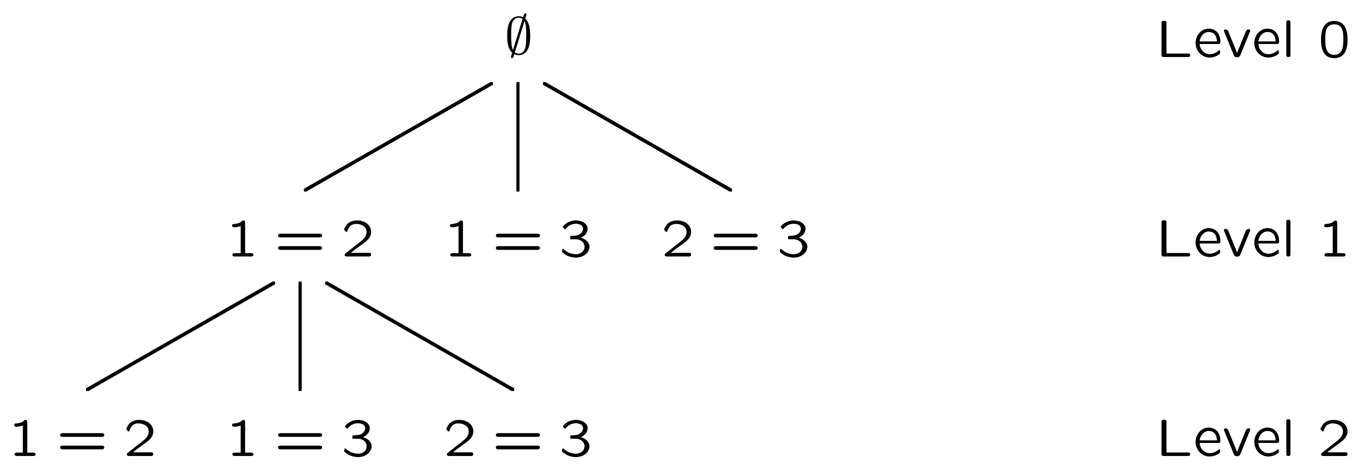
If not rejected, then $u_i u_j^{-1}$ is added to a (partial) set of generators for H and the search continues:

	x_1	x_2	x_3	x_1^{-1}	x_2^{-1}	x_3^{-1}
1	2	3	4	5		
2				1		
3					1	
4						1
5	1					
⋮						

Level	Coincidence	Additional generator
1	$1 = 2$	x_1^{-1}
	$1 = 3$	x_2^{-1}
	$2 = 3$	$x_1 x_2^{-1}$
	⋮	⋮
	$1 = 5$	x_1
	$2 = 5$	x_1^2
	$3 = 5$	$x_2 x_1$
	⋮	⋮

Branching/backtrack process

Systematic enumeration of coincidences between cosets (and adding new generators for H) sets up a branching process:



A backtrack search will terminate (given sufficient time and memory), by Schreier's theorem: every subgroup of finite index in a finitely-generated group is itself finitely-generated.

Example:

Let $G = \langle\, x, y \,|\, x^2 = y^3 = 1 \,\rangle$, which is the modular group, and look for subgroups of index up to 4. We get these:

#	Coincidences	Index	Generators
1	$1=2,\ 1=2$	1	$x,\ y$
2	$1=2,\ 2=4,\ 3=4$	3	$x,\ yxy^{-1},\ y^{-1}xy$
3	$1=2,\ 2=4,\ 4=5$	4	$x,\ yxy^{-1},\ y^{-1}xy^{-1}xy$
4	$1=2,\ 3=4$	3	$x,\ y^{-1}xy^{-1}$
5	$1=3,\ 2=3$	2	$y^{-1},\ xy^{-1}x$
6	$1=3,\ 4=5$	4	$y^{-1},\ xy^{-1}xy^{-1}x$

Low index normal subgroups

Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations.

Alternatively, the (standard) low index subgroups method can be adapted to produce only normal subgroups.

A new method was developed 14 years ago by Derek Holt and his student, which systematically enumerates the possibilities for the composition series of the factor group G/K, for any normal subgroup K of small index in G.

This method has produced the automorphism groups of lots of symmetric structures (incl. graphs, maps & polytopes).

Summary:

- Standard ‘Low Index Subgroups’ algorithm
- New variant for finding normal subgroups only
- These two methods can help find all small degree transitive permutation representations and all small quotients of a given finitely-presented group.

Low index subgroup methods

M.D.E. Conder & P. Dobcsányi, Applications and adaptations of the low index subgroups procedure, *Mathematics of Computation* 74 (2005), 485–497.

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D.F. Holt, B. Eick & E.A. O'Brien, *Handbook of Computational Group Theory*, CRC Press, 2005.

D. Firth, *An algorithm to find normal subgroups of a finitely presented group up to a given index*, PhD Thesis, University of Warwick, 2005.

C.C. Sims, *Computation with Finitely-Presented Groups* (Cambridge Univ. Press, 1994).