Combinatorial limits and their applications in extremal combinatorics

Part 2

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Chernoff Bound

- independent zero-one random variables $Z_1, \ldots, Z_n$
  concentration of $X = Z_1 + \cdots + Z_n$ around $\mathbb{E}X$
- Chernoff Bound
  $\mathbb{P}\left(|X - \mathbb{E}X| \geq \delta\mathbb{E}X\right) \leq 2e^{\frac{-\delta^2 \mathbb{E}X}{3}}$
  for every $\delta \in [0, 1]$
- if each Z_i is one with probability p, then
  $\mathbb{P}\left(|X - pn| \geq \delta pn\right) \leq 2e^{\frac{-\delta^2 pn}{3}}$

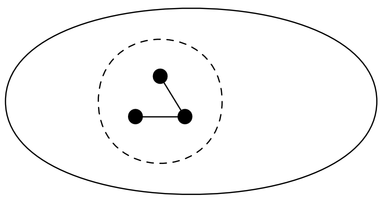
Martingales

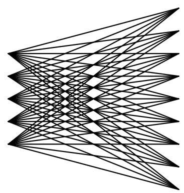
- martingale is a sequence of random variables X_n
  $\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = X_n$ for every $n \in \mathbb{N}$
- Azuma-Hoeffding inequality
  suppose that $\mathbb{E}X_n = X_0$ and $|X_n - X_{n-1}| \leq c_n$
  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$
- Doob's Martingale Convergence Theorem (corr.)
  if $|X_n| < K$, then $X_n \to X$ almost everywhere

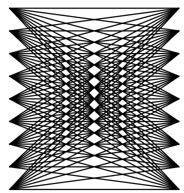
Questions?

Dense graph convergence

- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent
  if $d(H, G_n)$ converges for every H
- examples of convergent sequences:
  complete and complete bipartite graphs K_n and $K_{\alpha n,n}$
  Erdős-Rényi random graphs $G_{n,p}$

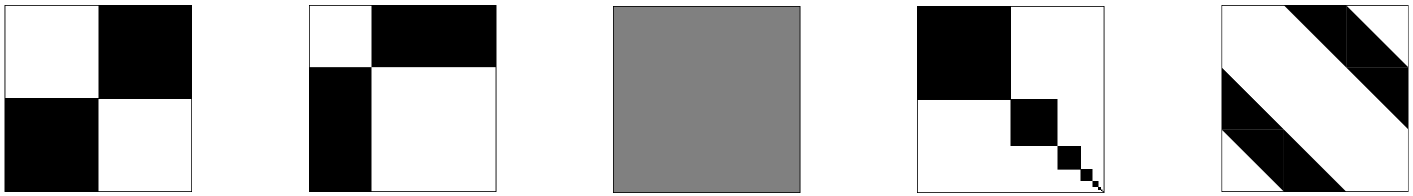






LIMIT OBJECT: GRAPHON

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$



Questions?

Graphons as limits

- Uniqueness of a graphon representing a sequence.
- Is every graphon a limit of convergent sequence?
- Does every convergent sequence have a limit?

W-random graphs converge

- A sequence of W-random graphs with increasing orders converges with probability one.
- fix $n \in \mathbb{N}$, a graph H and a graphon W
- X_i = exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them
- apply Azuma-Hoeffding inequality with $c_i = n^{|H|-1}$
  $\mathbb{P}\left(|X_n - X_0| \geq \varepsilon n^{|H|}\right) \leq 2e^{-\varepsilon^2 n/2}$

  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$

W-random graphs converge

- A sequence of W-random graphs with increasing orders converges with probability one.
- X_i = exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them
  $\mathbb{P}\left(\frac{|X_n - X_0|}{n^{|H|}} \geq \varepsilon\right) \leq 2e^{-\varepsilon^2 n/2}$
- the sum of $2e^{-\varepsilon^2 n/2}$ is finite for every $\varepsilon > 0$
- Borel-Cantelli $\Rightarrow$ the sequence converges with prob. one
- $X_0 \approx \frac{d(H,W)n^{|H|}}{|H|!}$ $\Rightarrow$ the graphon W is its limit

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Graph regularity

- Frieze-Kannan regularity, Szemerédi regularity
- $\forall \varepsilon > 0 \; \exists K_\varepsilon$ such that every graph G has an ε-regular equipartition $V_1, \ldots, V_k$ with $k \leq K_\varepsilon$
  $||V_i| - |V_j|| \leq 1$ for all i and j
- equipartition $V_1, \ldots, V_k \to$ density matrix $A_{ij} = \frac{e(V_i, V_j)}{|V_i||V_j|}$
- $\forall \delta > 0, H \; \exists \varepsilon > 0$ such that the density matrix of an ε-regular partition determines $d(H, G)$ upto an δ-error
- the lemma holds with prepartitions

Existence of limit graphon

- fix a convergent sequence G_i, $i \in \mathbb{N}$, of graphs
- set $\varepsilon_j = 2^{-j}$ and fix ε_1-regular partition of G_i
  fix ε_{j+1}-regular partition refining the ε_j-regular one
- take a subsequence G_i' of G_i such that all but finitely many ε_j-regular partitions have the same num. parts
- let A^{ij} be the density matrix for G_i and ε_j
- take a subsequence G_i'' of G_i' such that
  A^{ij} coordinate-wise converge for every j

Existence of limit graphon

- a convergent sequence G_i, density matrices A^{ij}
  let A^j be the coordinate-wise limit of A^{ij}
- interpret A^j as a random variable on $[0,1]^2$ and
  apply Doob's Martingale Convergence Theorem
  in this way, we obtain a graphon W
- relate $d(H, W)$ to the density of H based on A^j

Questions?

Other combinatorial objects

- dense graph convergence
  convergence of substructure densities
- extendable to other combinatorial structures
  directed graphs, edge-colored graphs, hypergraphs
  partial orders, permutations, ...
- sparse graph convergence
  Benjamini-Schramm convergence, local-global conv.,
  partition convergence, large deviation convergence, ...

Permutations

- permutation of order n: order on numbers $1, \ldots, n$
  subpermutation: $4\underline{53}21\underline{6} \quad \longrightarrow \quad 213$
- density of a permutation π in a permutation Π:

$$d(\pi, \Pi) = \frac{\#\text{ subpermutations of } \Pi \text{ that are } \pi}{\#\text{ all subpermutations of order } \pi}$$

- $(\Pi_j)_{j \in \mathbb{N}}$ convergent if $\exists \lim\limits_{j \to \infty} d(\pi, \Pi_j)$ for every π

Representation of a limit

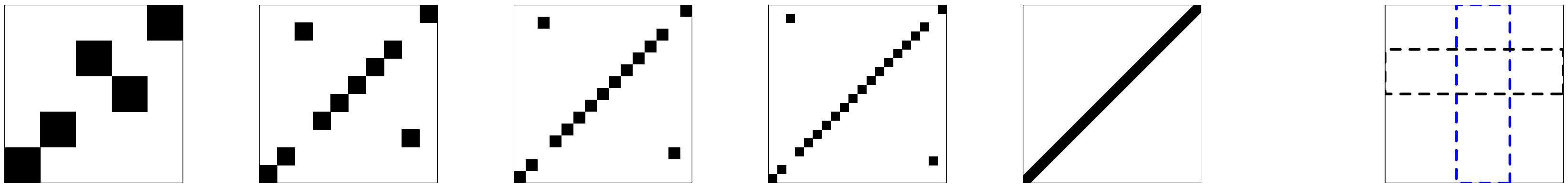
- probability measure μ on $[0,1]^2$ with unit marginals

  $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a$

  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio

- μ-random permutation

  choose n random points, x- and y-coordinates



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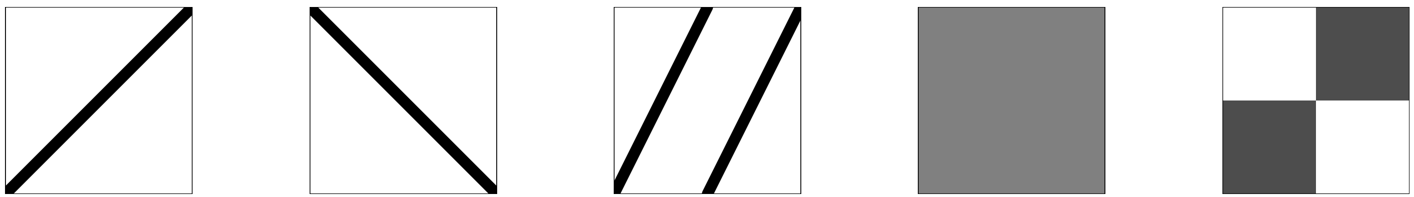

Questions?

Thank you for your attention!