Three Open Problems

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The first open problem gives a potential characterization of the Q-**polynomial symmetric association schemes**.

Motivation. Assume that $\Gamma = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme that is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents.

Let $V = \mathbb{C}X$ denote the **standard module** of Γ.

Problem 1 motivation, cont.

Abbreviate

$$V_i = E_i V \qquad (0 \le i \le d).$$

For $0 \le n \le d$ define

$$V_1^{\circ n} = \mathrm{Span}\big(V_1 \circ V_1 \circ \cdots \circ V_1\big) \qquad (n \text{ copies})$$

where $\circ$ is **entry-wise multiplication**.

We interpret $V_1^{\circ 0} = \mathrm{Span}\{\mathbf{1}\}$, where $\mathbf{1}$ is the all-ones vector in V.

Problem 1 motivation, cont.

Since the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial, we have

$$V_0 + V_1 + \cdots + V_i = V_1^{\circ 0} + V_1^{\circ 1} + \cdots + V_1^{\circ i}$$

for $0 \leq i \leq d$.

We now reverse the logical direction...

Problem 1 statement

Problem

Let $\Gamma = (X, \mathcal{R})$ denote an undirected, connected, regular graph, with adjacency matrix A and standard module V.

Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A.

Assume that

$$V_0 + V_1 + \cdots + V_i = V_1^{\circ 0} + V_1^{\circ 1} + \cdots + V_1^{\circ i}$$

for $0 \le i \le d$.

Show that A generates the **Bose-Mesner algebra of a symmetric association scheme on** X (or find counterexamples).

Problem 2

The second open problem is about the **subconstituent algebra of a distance-regular graph**.

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph, with diameter d, adjacency matrix A, and standard module V.

Problem 2, cont.

For $x \in X$ let $\hat{x}$ denote the vector in V that has x-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{x} | x \in X\}$ form a basis for V.

Definition

For $x \in X$ and $0 \leq i \leq d$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad y \in X.$$

Definition

For $x \in X$ let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and $\{E_i^*\}_{i=0}^d$.

We call T the **subconstituent algebra** of Γ with respect to x.

Problem 2, cont.

For the rest of this section, fix distinct $x, y \in X$.

Definition

Let $T(x, y)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $T(x)$ and $T(y)$.

We call $T(x, y)$ the **subconstituent algebra of Γ with respect to** x, y.

The following lemmas are routine.

Lemma

The standard module V is a direct sum of irreducible $T(x,y)$-modules.

Lemma

There exists a unique irreducible $T(x,y)$-module that contains the all-ones vector $\mathbf{1}$. This module is $T(x,y)\mathbf{1}$.

Problem 2, cont.

Lemma

We have

$$T(x)\hat{y} \subseteq T(x,y)\mathbf{1}, \qquad T(y)\hat{x} \subseteq T(x,y)\mathbf{1}.$$

Problem 2 statement

Problem

Assume that Γ *is Q-polynomial. Show that*

$$T(x)\hat{y} = T(x,y)\mathbf{1} = T(y)\hat{x}.$$

More generally, decompose $T(x,y)\mathbf{1}$ *into a direct sum of irreducible* $T(x)$*-modules and a direct sum of irreducible* $T(y)$*-modules. We expect that in each direct sum, the summands are mutually nonisomorphic.*

Problem 3

The third open problem is about **some central elements in the subconstituent algebra** of a distance-regular graph.

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph, with adjacency matrix A and standard module V.

Let E denote a Q-polynomial primitive idempotent of Γ.

Write

$$E = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i$$

where A_i is the ith distance matrix of Γ.

The scalars $\{\theta_i^*\}_{i=0}^d$ form the **dual eigenvalue sequence** of Γ with respect to E.

For $x \in X$, the **dual adjacency matrix** $A^* = A^*(x)$ is defined by

$$A^* = \sum_{i=0}^{d} \theta_i^* E_i^*.$$

The subconstituent algebra $T = T(x)$ is generated by A, A^*.

The matrices A, A^* satisfy the **tridiagonal relations**

$$
\begin{aligned}
0 &= [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],\\
0 &= [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A],
\end{aligned}
$$

for appropriate real scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$.

Problem 3, cont.

We recall the thin condition.

An irreducible T-module W is **thin** whenever $E_i^* W$ has dimension at most one for $0 \leq i \leq d$.

In this case, $E_i W$ has dimension at most one for $0 \leq i \leq d$.

If W is thin then on W,

$$\begin{aligned}
&A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* \\
&\qquad = \gamma^* A^2 + \omega A + \eta I, \\
&A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A \\
&\qquad = \gamma A^{*2} + \omega A^* + \eta^* I,
\end{aligned}$$

where ω, η, η^* are appropriate real scalars that depend on W.

These are the **Askey-Wilson relations**.

Problem 3, cont.

Lemma (Worawannotai 2012)

Assume that every irreducible T-module is thin. Then there exist **central elements** $\Omega, G, G^* \in T$ *such that*

$$
\begin{aligned}
&A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* \\
&\qquad = \gamma^* A^2 + \Omega A + G, \\
&A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A \\
&\qquad = \gamma A^{*2} + \Omega A^* + G^*.
\end{aligned}
$$

Problem 3 statement

Problem

Assume that every irreducible T-module is thin.

(i) *Find the* **entries** *of* Ω, G, G^* *(for some examples).*

(ii) *We conjecture that for* $y, z \in X$ *the* (y,z)*-entry of* Ω, G, G^* *are all zero unless* $\partial(y,z) \leq 2$.

(iii) *Find the* **combinatorial significance** *of* Ω, G, G^*.

Problem 3 comments

Chalermong Worawannotai (2012) has results about Problem 3 for the **dual polar graphs**.

Ian Seong (2025) has results about Problem 3 for the **Grassmann graph** $J_q(n, d)$.

THE END